PHYS 301 Last Time Oct. 18, 2024

3.3 Separation of Variables (Cartesian Coords)

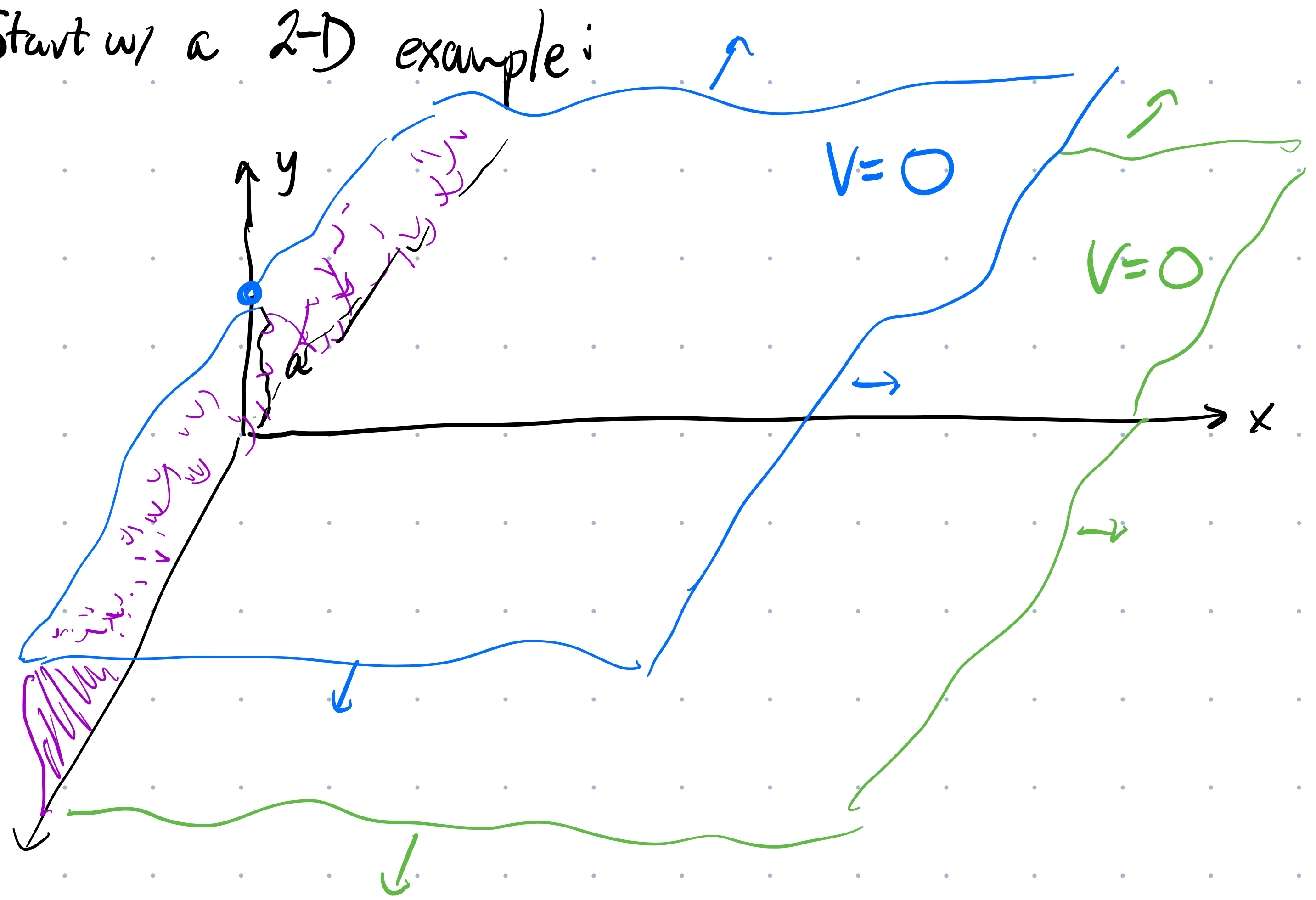
Attempt to solve the partial diff. eq'n

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \text{ directly}$$

assuming we can write

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

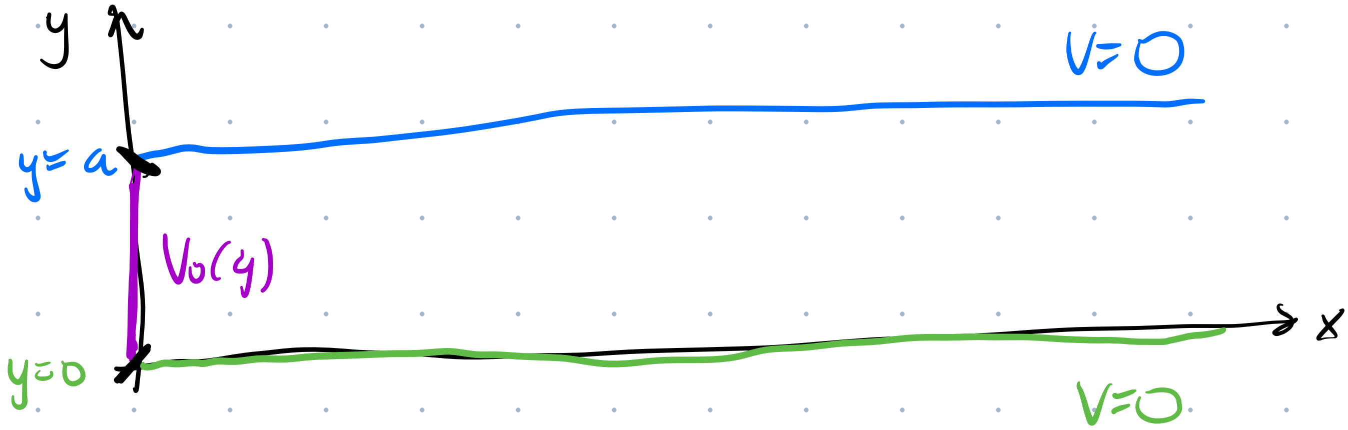
Start w/ a 2-D example:



There is a strip along z-dir'n that is at a potential $V = V_0(y)$

The strip extends from $y=0$ to $y=a$

Side View



boundary conditions:

(i) $V=0$ when $y=0$
(ii) $V=0$ when $y=a$
(iii) $V=0$ when $x \to \infty$
(iv) $V=V_0(y)$ when $x=0$

Last time, found that the sol'n below satisfies b.c.'s (i), (ii) & (iii)

$$\boxed{V(x,y) = C' e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right)}$$

This "sol'n" does not currently satisfy b.c. (iv)
Still have to consider $V(x=0) = V_0(y)$

However, any linear combination of $e^{-kx} \sin ky$ is also a sol'n to $\nabla^2 V = 0$.

Maybe we can construct

$$V(x,y) = \sum_{n=1}^{\infty} b_n e^{-kx} \sin ky$$

Require:

$$V(0,y) = \underbrace{\sum_{n=1}^{\infty} b_n \sin ky}_{\text{Fourier series}} = \underbrace{V_0(y)}_{\text{b.c.}}$$

Can select coefficients b_n s.t. $V(0,y)$ matches any $V_0(y)$.

To go further, we need to specify form of $V_0(y)$

Select $V_0(y) = V_0$ (const).

Just like for Fourier series, b_n is found by evaluating

$$b_n = \frac{2}{a}\int_0^a \underbrace{V(0,y)}_{V_0} \sin\left(\frac{n\pi}{a}y\right) dy$$

$$= \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi}{a}y\right) dy$$

$$= -\frac{2V_0}{\cancel{a}} \frac{\cancel{a}}{n\pi} \cos\left(\frac{n\pi}{a}y\right)\Bigg|_0^a$$

$$b_n = -\frac{2V_0}{n\pi}\left[\underbrace{\cos(n\pi)}_{(-1)^n} - 1\right]$$

$$b_n = -\frac{2V_0}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{\begin{aligned} &b_n = 0 \quad \text{for } n \text{ even} \\ &b_n = \frac{4V_0}{n\pi} \quad n \text{ odd} \end{aligned}}$$

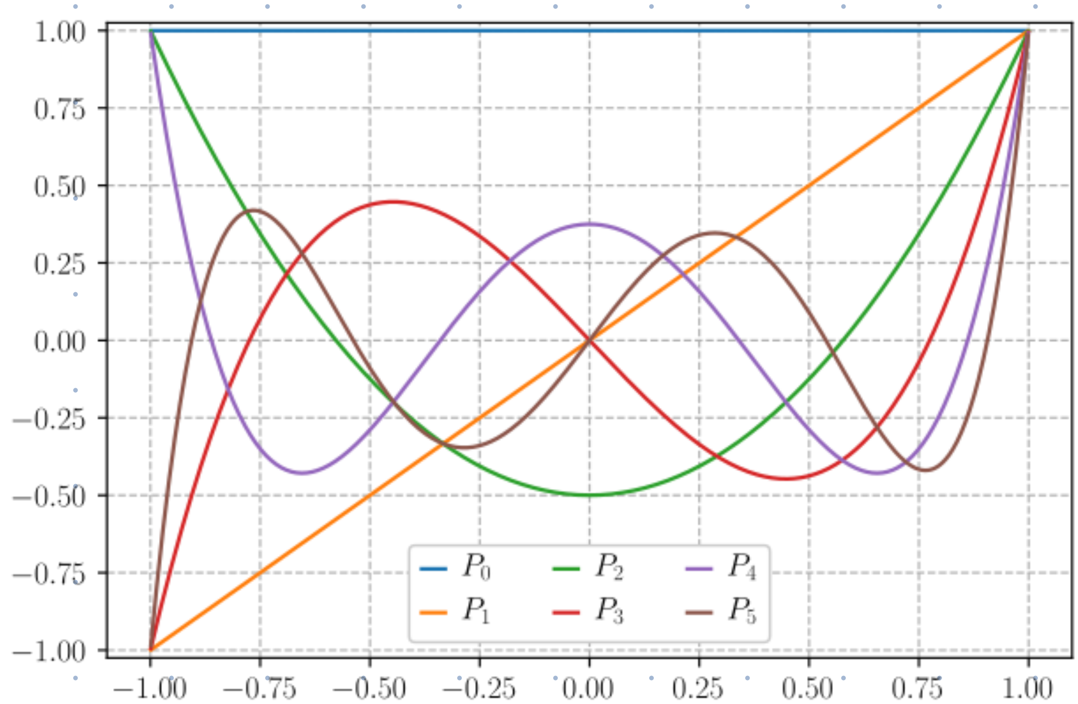
The final sol'n for $V(x,y)$ is:

$$\boxed{V(x,y) = \sum_{\substack{n \text{ odd} \\ (1,3,5,7,\ldots)}}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right)}$$

Legendre Polynomials $P_\ell(x)$

will be interested in region $-1<x<1$

ℓ	$P_\ell(x)$	even/odd	$P_\ell(1)$	$P_\ell(-1)$
0	1	even	1	1
1	x	odd	1	-1
2	$\frac{1}{2}(3x^2-1)$	even	1	1
3	$\frac{1}{2}(5x^3-3x)$	odd	1	-1
4	$\frac{1}{8}(35x^4-30x^2+3)$	even	1	1
5	$\frac{1}{8}(63x^5-70x^3+15x)$	odd	1	-1
⋮	⋮	⋮	⋮	⋮



Legendre Polynomials arise as solutions to the following differential eq'n:

$$(1-x^2)\frac{d^2y(x)}{dx^2} - 2x\frac{dy(x)}{dx} = -l(l+1)y(x) \quad \text{\#}$$

where $l = 0, 1, 2, \ldots$ We're interested $-1 < x < 1$

Seems arbitrary, but we will encounter # when solving $\nabla^2 V = 0$ in spherical coords using separation of variables.

Consider $l=0$ case of #

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0$$

$y=1$ is a sol'n

$y = P_0(x) = 1$

case $l=1$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = -2y$$

$y = x$ is a sol'n

$y = P_1(x) = x$

case $\ell=2$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = -6y$$

Sol'n not obvious, try $y = P_2(x) = \frac{1}{2}(3x^2-1)$

$$\frac{dy}{dx} = 3x \qquad \frac{d^2y}{dx^2} = 3$$

$$3(1-x^2) - 2x(3x) = -6\,\tfrac{1}{2}(3x^2-1)$$

$$\cancel{3} - \cancel{3x^2} - \cancel{6x^2} = -\cancel{9x^2} + \cancel{3} \quad \checkmark$$

$$y = P_2(x) = \tfrac{1}{2}(3x^2-1)$$

The Legendre Polynomials can be generated using the so-called Rodrigues Formular

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^{(\ell)}}{dx^{(\ell)}}(x^2-1)^\ell$$

$$P_0 = 1 \quad \checkmark$$

$$P_1 = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{2}\,2x = x \quad \checkmark$$

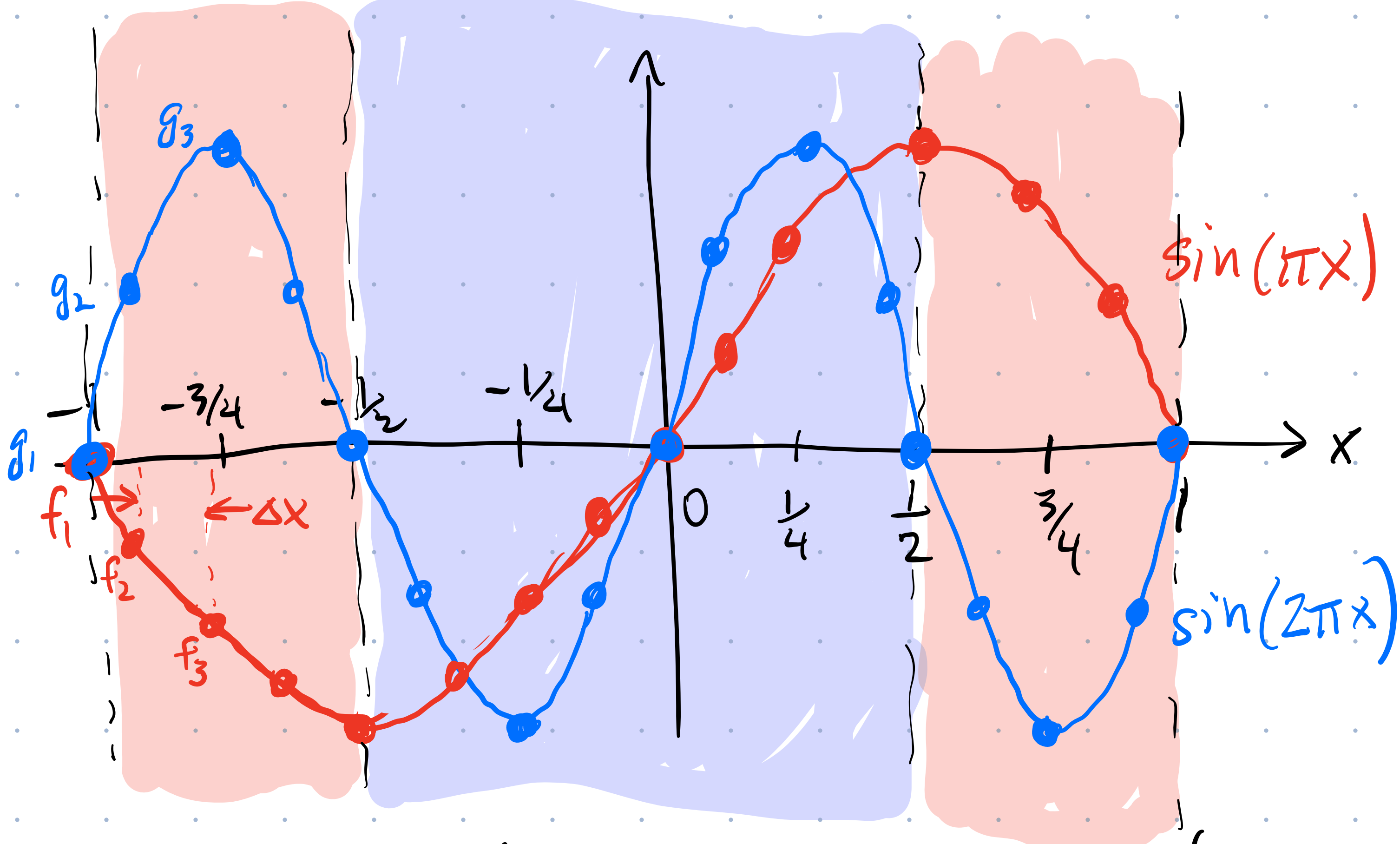
$$P_2 = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2} \left(x^2 - 1 \right)^2$$

$$= \frac{1}{8} \frac{d}{dx} \left[2(x^2-1) 2x \right]$$

$$= \frac{1}{2} \frac{d}{dx} \left[x^3 - x \right] = \frac{1}{2} \left[3x^2 - 1 \right] \checkmark$$

$$\vdots$$

Aside: Let's consider $\sin(\pi x)$ & $\sin(2\pi x)$



$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix} \qquad \vec{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_N \end{pmatrix}$$

$$\Delta x = \frac{b-a}{N}$$

$a < x < b$ domain

N: no. of samples

Try to think about fcns $f(x)$ as vectors w/ components. Do this by sampling fcns at evenly-spaced interval Δx.

The components of the vectors are given by:

$$f_1 = f(x_1)$$
$$f_2 = f(x_2)$$
$$\vdots$$

Want to construct something similar to a dot product, but for fcns.

If have 3-D vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$

Then $\vec{v} \cdot \vec{u} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \sum_{i=x,y,z} v_i u_i$

The analogous "inner product" for fcns might be:

$$\begin{pmatrix} f_1 & f_2 & f_3 & \ldots & f_N \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_N \end{pmatrix} = \sum_{i=1}^{N} f_i g_i$$

$$\therefore \langle \vec{f}, \vec{g} \rangle = \cancel{\sum_{i=1}^{N} f(x_i)} g(x_i)$$

Problem w/ this definition of inner product is that as N increases, the value of the sum is not stable.

If this issue by multiplying by $\Delta x = \frac{b-a}{N}$

$$\langle \vec{f}, \vec{g} \rangle = \sum_{i=1}^{N} f(x_i) g(x_i) \Delta x$$

In limit $\Delta x \to 0$

$$\boxed{\langle \vec{f}, \vec{g} \rangle = \int_a^b f(x) g(x) \, dx}$$

If $f(x)$ & $g(x)$ are similar, find $\langle f, g \rangle$ large.

If " " " " very diff. " " small.

For $f = \sin(\pi x)$ & $g = \sin(2\pi x)$

$\langle f, g \rangle = 0 \Rightarrow$ $f(x)$ & $g(x)$ are orthogonal.

Check

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} \underbrace{\sin(m\pi x)}_{f(x)} \underbrace{\sin(n\pi x)}_{g(x)} \, dx = 0$$

$\forall$ n, m with $n \neq m$

n, m integers

$m \neq n$.

$\therefore$ $\sin(n\pi x)$ defines a set of orthogonal fcns.

Consider 3-D vector space. A set of orthogonal vectors in that space are $\hat{x}, \hat{y}, \hat{z}$. It is possible to contruct any vector $\vec{V}$ in that space in terms of components along $\hat{x}, \hat{y}, \hat{z}$.

$$\vec{V} = V_x\,\hat{x} + V_y\,\hat{y} + V_z\,\hat{z}$$

$$= \underbrace{(\vec{V}\cdot\hat{x})}_{V_x}\,\hat{x} + (\vec{V}\cdot\hat{y})\,\hat{y} + (\vec{V}\cdot\hat{z})\,\hat{z}$$

In a similar way, we can write any fcn f(x) in terms of "components" aligned w/ the orthogonal basis fcns $\sin(n\pi x)$

$$f(x) = \langle f(x), \sin \pi x\rangle \sin \pi x$$
$$+ \langle f(x), \sin 2\pi x\rangle \sin 2\pi x$$
$$+ \langle f(x), \sin 3\pi x\rangle \sin 3\pi x$$
$$+ \dots$$

call $b_n = \langle f(x), \sin n\pi x\rangle$

$$\boxed{\begin{gathered} b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx \\ f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \end{gathered}}$$

Legendre Polynomials are another set of orthogonal fcns.

$$f(x) = \overbrace{\langle f(x), P_0(x) \rangle}^{c_0} P_0(x) + \overbrace{\langle f(x), P_1(x) \rangle}^{c_1} P_1(x)$$

$$+ \cdots + \underbrace{\langle f(x), P_\ell(x) \rangle}_{c_\ell} P_\ell(x)$$

$$\boxed{\begin{aligned} f(x) &= \sum_{\ell=0}^{\infty} c_\ell P_\ell(x) \\ \text{where} \quad c_\ell &= \int_{-1}^{1} f(x) P_\ell(x)\, dx \end{aligned}}$$