PHYS 301 Oct. 9, 2024

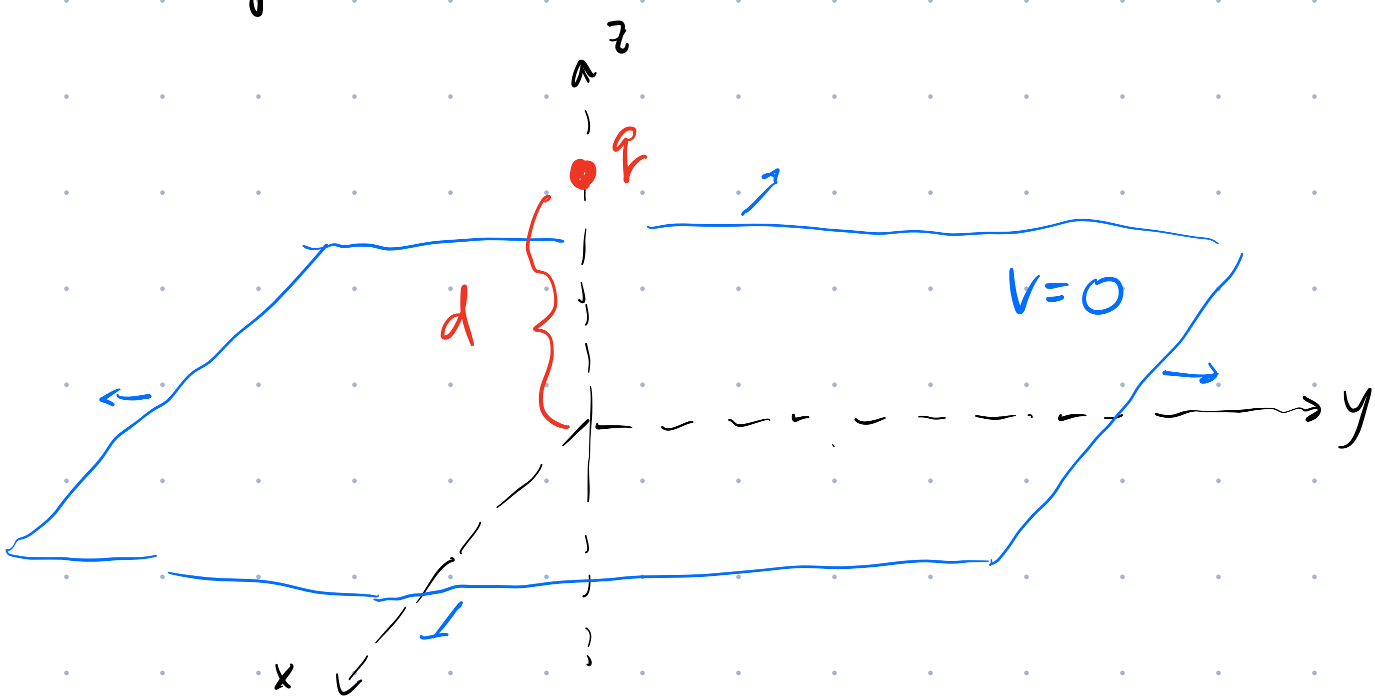
Last Time:

First Uniqueness Theorem: The sol'n to $\nabla^2 V = 0$ in a volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary of surrounding surface S.

→ In other words, if we can find $V(x, y, z)$ that satisfies $\nabla^2 V = 0$ & the given boundary conditions, it is guaranteed to be the one & only solution to our problem.

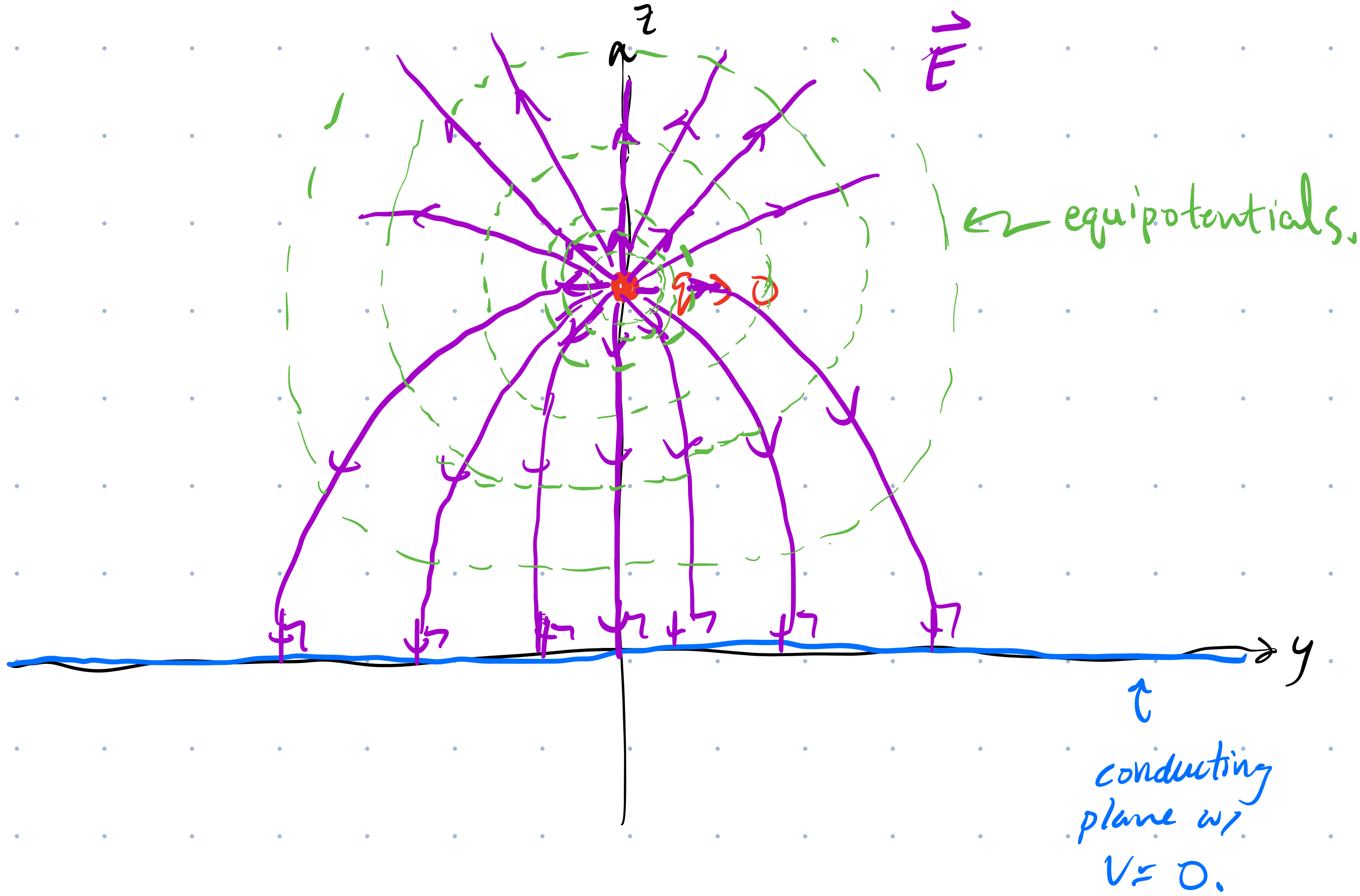
Griffiths 3.2 Method of Images

Consider a pt. charge q a dist. $z = d$ above a conducting sheet at $z = 0$. Conducting sheet is held at $V = 0$ (grounded). What is $V(x,y,z)$ in the region $z > 0$.

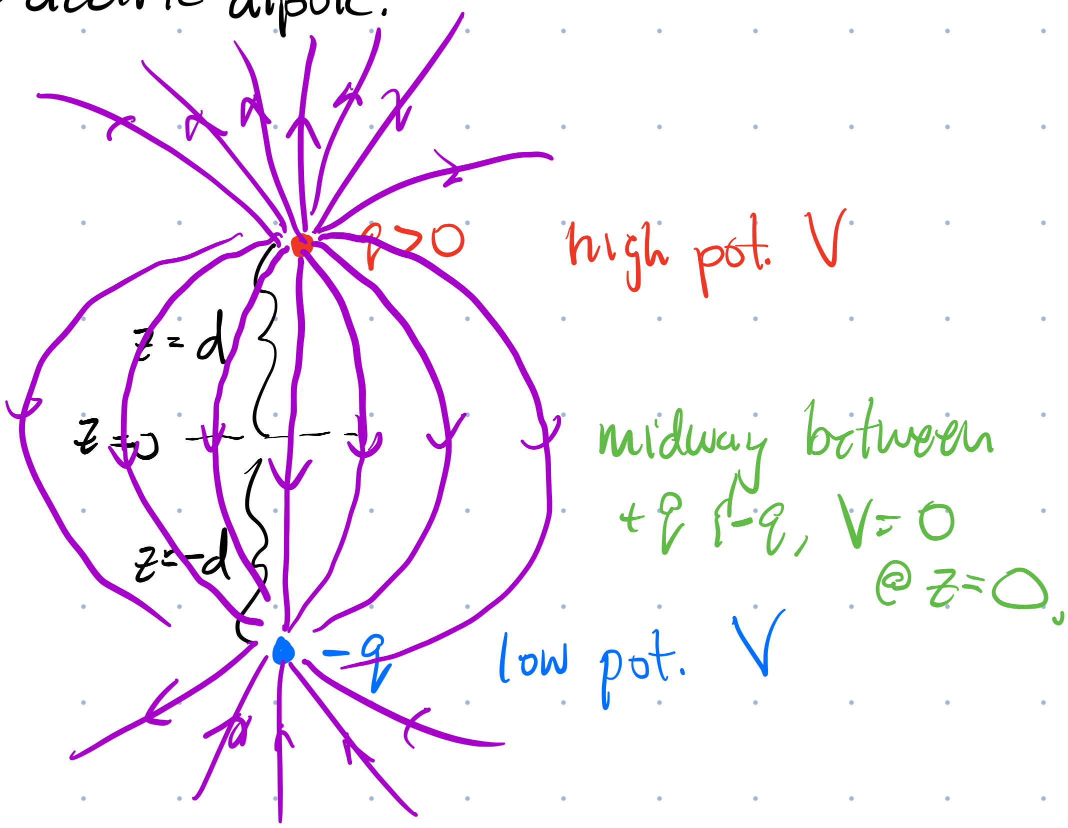


Before trying to solve this problem, let's sketch what we think $\vec{E}$ & V will look like in region $z > 0$.

Side View



Looks like half of the $\vec{E}$ & V that results from an electric dipole.

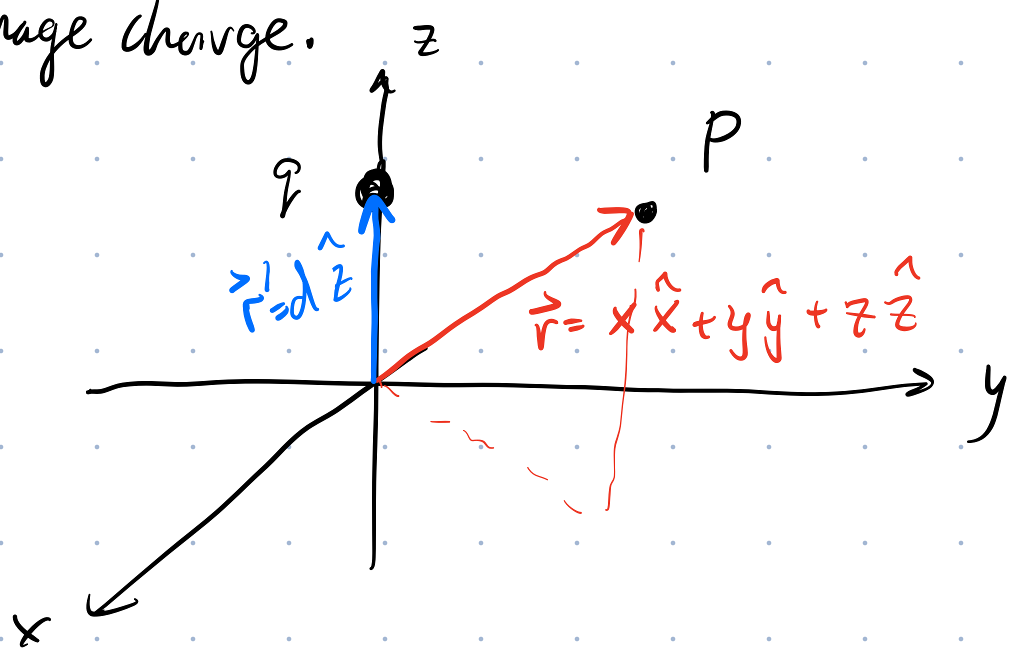


In our original problem w/ infinite plane, the boundary conditions (b.c.):

- far away from q, $V=0$
- everywhere where $z=0$, $V=0$.

$\therefore$ Dipole config. satisfies b.c. . Since V of pt. charges must satisfy Laplace's eq'n, the two configurations above must have exactly the $V(x,y,z)$ in the region $z>0$. $\Rightarrow$ First uniqueness theorem.

Notice that we essentially treated gnded plane as a mirror while also changing the sign of the image charge.



For pt. charge $V_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{R}_+}$

$$\vec{\mathscr{R}}_+ = \vec{r} - \vec{r}\,' = x\,\hat{x} + y\,\hat{y} + (z-d)\,\hat{z}$$

$\mathscr{R}_+ = \sqrt{x^2+y^2+(z-d)^2}$ for pos. charge.

$$V_- = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{\mathscr{R}_-}$$

$$\mathscr{R}_- = \sqrt{x^2+y^2+(z+d)^2}$$

The total potential in $z > 0$ region of our original problem is given by:

$$V = V_+ + V_-$$

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

valid for $z > 0$

Now that V is known, it can be used to find the surface charge induced on ground plane.

Know that just above a conductor $\vec{E}$ is $\perp$ to surface.

$$E_\perp = E_z = \frac{\sigma}{\varepsilon_0}$$

$$\therefore -\frac{\partial V}{\partial z} = \frac{\sigma}{\varepsilon_0}$$

$$\therefore \sigma = -\varepsilon_0 \frac{\partial V}{\partial z}$$

Consider $\frac{\partial}{\partial z}\left[x^2+y^2+(z\pm d)^2\right]^{-1/2}$

$$= \frac{-\frac{1}{\cancel{2}}\,\cancel{2}(z\pm d)}{\left[x^2+y^2+(z\pm d)^2\right]^{3/2}}$$

$$\therefore \sigma = \frac{\cancel{\varepsilon_0} q}{4\pi \cancel{\varepsilon_0}} \left[\frac{z-d}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}} \right]$$

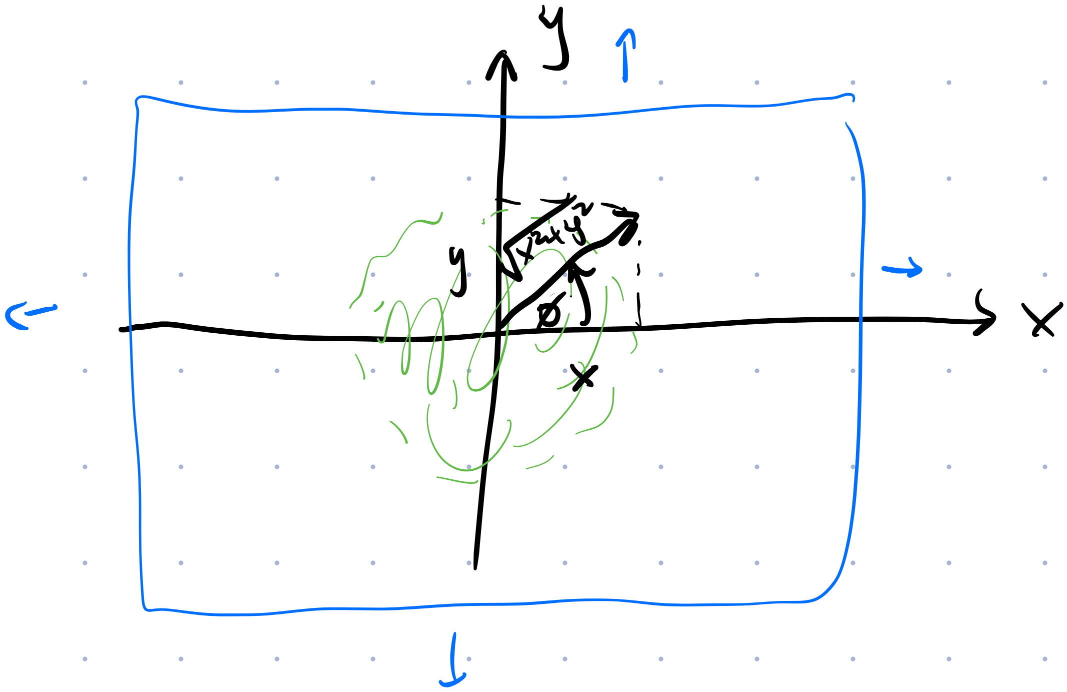
$$= \frac{+q}{4\pi} \left[\frac{z-d}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{-z-d}{\left[x^2+y^2+(z+d)^2\right]^{3/2}} \right]$$

Our conducting plane sits at $z=0$.

$$\sigma = \frac{q}{4\pi} \left[\frac{-d}{\left[x^2+y^2+d^2\right]^{3/2}} - \frac{d}{\left[x^2+y^2+d^2\right]^{3/2}} \right]$$

$$= \frac{-2qd}{4\pi \left[x^2+y^2+d^2\right]^{3/2}}$$

xy-plane



Switch to polar coord. $x^2+y^2=r^2$

$0<r<\infty$

$0<\phi<2\pi$

Integrate σ over area of plane to find the total induced charge.

$$q_{ind.} = \int_{plane} \sigma \, da$$

$$q_{ind} = -\frac{qd}{2\pi} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \int_{r=0}^{\infty} \frac{1}{[r^2+d^2]^{3/2}} r\,dr$$

$$u = r^2 + d^2 \qquad du = 2r\,dr \qquad r\,dr = \frac{du}{2}$$

$$r = 0 \Rightarrow u = d^2$$

$$r \to \infty \Rightarrow u \to \infty$$

$$q_{ind} = -\frac{qd}{2} \int_{u=d^2}^{\infty} u^{-3/2}\,du$$

$$= qd \left. \frac{1}{u^{1/2}} \right|_{d^2}^{\infty}$$

$$= qd \left[0 - \frac{1}{d} \right] \Rightarrow \boxed{q_{ind} = -q}$$

3.3 Separation of Variables (Cartesian Coords)

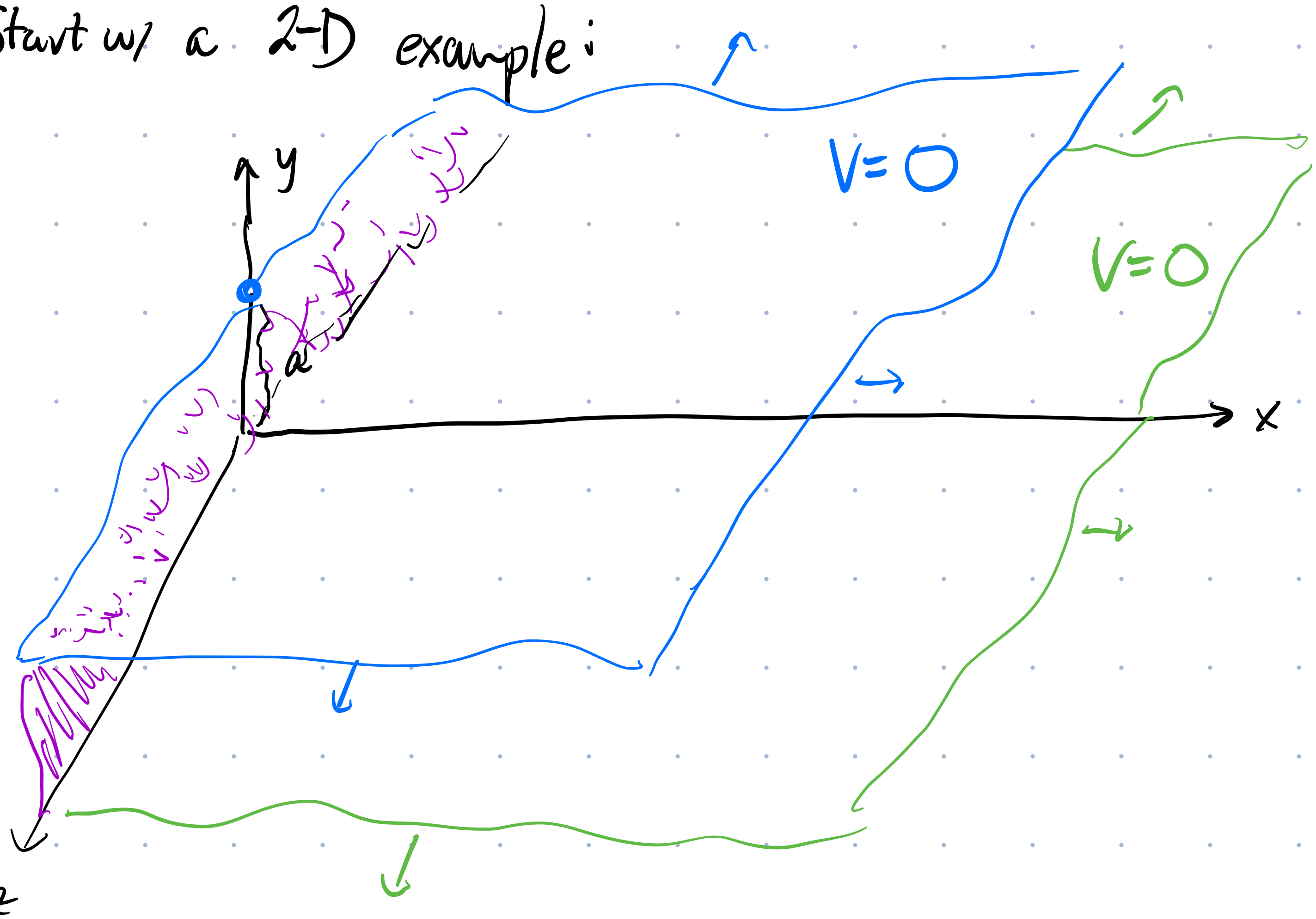
Attempt to solve the partial diff. eq'n

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{directly}$$

assuming we can write

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

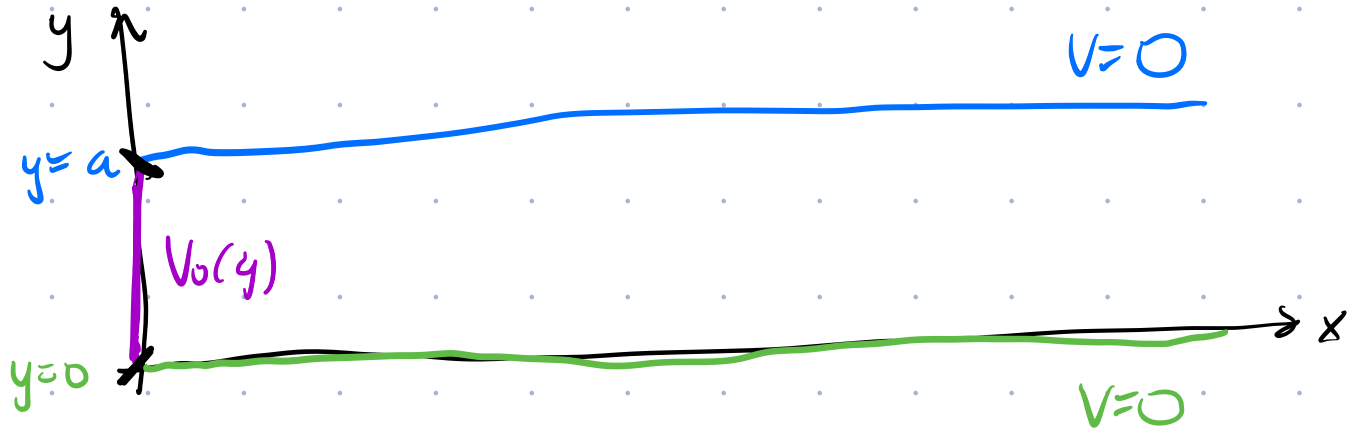
Start w/ a 2-D example:



Two parallel semi-infinite conducting sheets both at $V=0$. One sheet is in xz-plane. The other is at $y=a$.

There is a strip along z-dir'n that is at a potential $V = V_0(y)$
The strip extends from $y=0$ to $y=a$

Side View



By symmetry, expect V to be indep. of z.

$$V(x,y,z) \to V(x,y) = X(x)\, Y(y)$$

boundary conditions:

(i) $V=0$ when $y=0$
(ii) $V=0$ when $y=a$
(iii) $V=0$ when $x \to \infty$
(iv) $V = V_0(y)$ when $x=0$

Goal: Try to see if we can find $V(x,y) = X(x)\,Y(y)$ that satisfies both $\nabla^2 V = 0$ & all b.c.'s.

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}\left(X(x)\,Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)\,Y(y)\right) = 0$$

$$= Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

divide by XY.

$$\boxed{\therefore \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0} \quad (*)$$

must be the case that both

$\frac{1}{X}\frac{d^2X}{dx^2}$ & $\frac{1}{Y}\frac{d^2Y}{dy^2}$ are equal to constants

In fact, the constants are equal & opposite.

$$\frac{1}{X}\frac{d^2X}{dx^2} = C \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -C$$

in order to satisfy $(*)$

For convenience, choose to write the constant as $c = k^2$.

$$\therefore \quad \frac{1}{X}\frac{d^2X}{dx^2} = k^2 \Rightarrow \frac{d^2X}{dx^2} = k^2 X \quad ①$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \Rightarrow \frac{d^2Y}{dy^2} = -k^2 Y \quad ②$$

Separation of variables has allowed us to write of 2nd order partial diff. eq'n as a pair of ordinary diff. eq'ns.

Solutions ① $\quad X = Ae^{kx} + Be^{-kx}$

$$\frac{dX}{dx} = kAe^{kx} - kBe^{-kx}$$

$$\frac{d^2X}{dx^2} = k^2Ae^{kx} + k^2Be^{-kx}$$

$$= k^2 X \checkmark$$

② $\quad Y = C\sin ky + D\cos ky$

$$\frac{dY}{dy} = kC\cos ky - kD\sin ky$$

$$\frac{d^2Y}{dy^2} = -k^2C\sin ky - k^2D\cos ky = -k^2Y \checkmark$$

$$\therefore V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right)$$

If this sol'n is going to be valid, it must match all four b.c. (Physics)

Start w/ (iii) $V=0$ when $x \to \infty$

$$V(\infty, y) = \left(\underbrace{Ae^{k\infty}}_{\infty} + \cancel{Be^{-k\infty}}^{\,0}\right)\left(C\sin ky + D\cos ky\right) = 0$$

$\therefore$ Can only satisify b.c. (iii) if $A=0$.

Absorb const. B into C & D.

$$V(x,y) = e^{-kx}\left(C'\sin ky + D'\cos ky\right)$$

b.c. (i) $V=0$ when $y=0$

$$V(x,0) = e^{-kx}\left(C'(0) + D'(1)\right) = 0$$

require $D'=0$.

$$V(x,y) = C'e^{-kx}\sin ky$$

b.c. (ii) $V=0$ when $y=a$

$$V(x,a) = C'e^{-kx}\sin(ka) = 0$$

$ka = n\pi \qquad n = 1, 2, 3, \ldots$

$\hookrightarrow k = \frac{n\pi}{a}$

$$\boxed{V(x,y) = C'e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi}{a}y\right)}$$