PHYS 301 Oct. 7, 2024

Griffiths Ch. 3: Laplace's Eq'n.

Recall Poisson's Eq'n $\nabla^2 V = -\rho/\varepsilon_0$

In a region where $\rho = 0$, this reduces to Laplace's Eq'n

$$\nabla^2 V = 0$$

In Cartesian coords.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ Partial diff. eq'n.

3.1.2. Laplace's Eq'n in 1-D.

Strategy is to first work out some properties of the sol'ns to Laplace's eq'n in 1-D and then generalize to higher dimensions.

In 1-D

$$\frac{d^2V}{dx^2} = 0$$

$$\therefore \frac{dV}{dx} = m \quad (\text{a const}).$$

$$\Rightarrow V = mx + b \quad (\text{straight line})$$

The two integration const. are determined by boundary conditions.

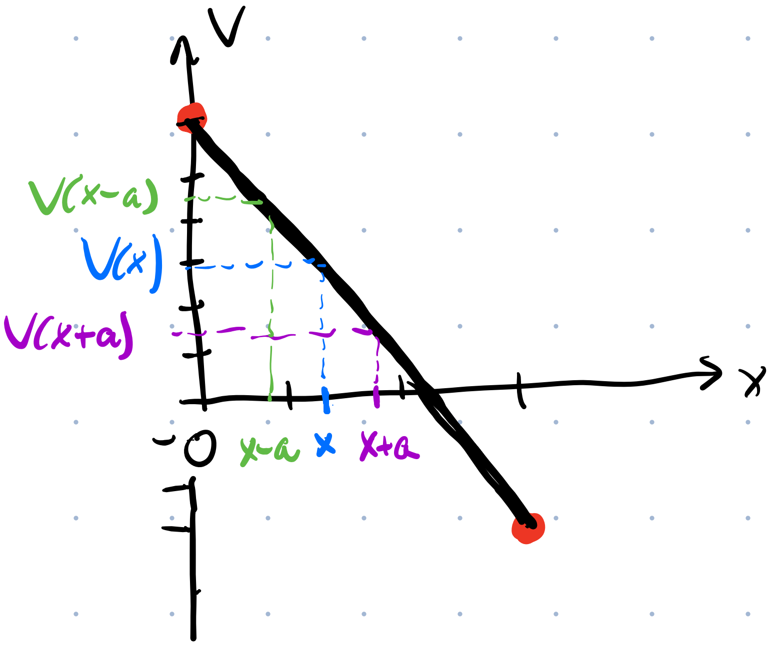
For example if $V(0) = 6$ & $V(3) = -3$

$$V(0) = m(0) + b = 6 \Rightarrow \underline{\underline{b = 6}}$$

$$V = mx + 6$$

$$V(3) = 3m + 6 = -3 \Rightarrow m = -3$$

$$\boxed{V = -3x + 6}$$



Note ①: V(x) has no local minima or maxima.
All extrema occur at the boundaries
($x=0$, $x=3$)

$$\frac{dV}{dx} = m \neq 0 \Rightarrow \text{no local min or max pts.}$$

$$\frac{d^2V}{dx^2} = 0$$

require $\frac{d^2V}{dx^2} < 0$ for maxima

$\frac{d^2V}{dx^2} > 0$ for minima.

Note ②: V(x) is the average of V(x+a) and V(x−a) for any a.

3.1.3 Laplace's Eq'n in 2-D

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Aside: Finding extrema of multivariable fcns

I'll write down procedure for finding local minima & maxima for 2-D fcns.

1. If $f(x,y)$, find all (x,y) pts that satisfy

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

2. Find the Hessian Matrix

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

3. For each pt. (x, y) from ①, get a minimum or maximum if $\det(H) > 0$.

$$\det(H) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$\therefore$ can only get minima or maxima if $\frac{\partial^2 f}{\partial x^2}$ & $\frac{\partial^2 f}{\partial y^2}$ have the same sign!

(If $\det(H) < 0$, saddle pt.
If $\det(H) = 0$, test inconclusive)

$$\nabla^2 f = 0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

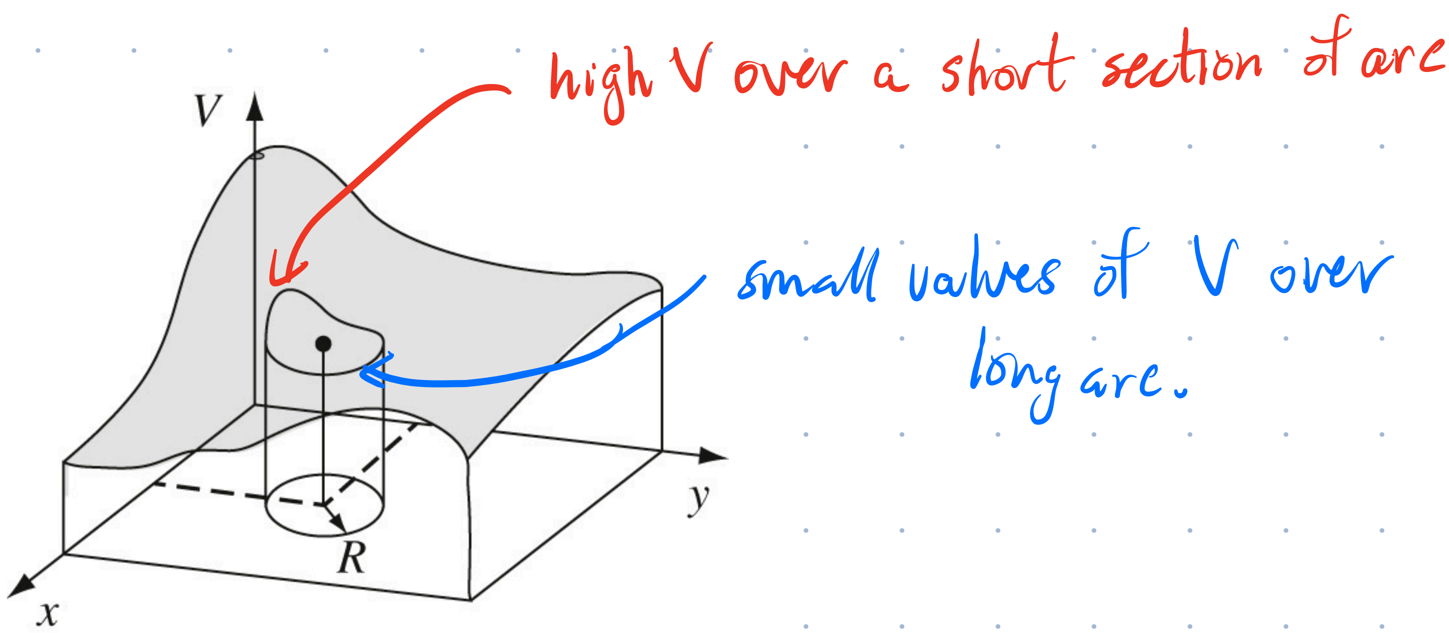
$\therefore$ $\frac{\partial^2 f}{\partial x^2}$ & $\frac{\partial^2 f}{\partial y^2}$ must have opp. sign.

$\therefore$ f cannot have local extrema.

This argument generalizes to 3-D

Note① $\boxed{\Rightarrow \text{In 1, 2, or 3-D sol'ns to Laplace's eq'n contain no extrema except at boundaries}}$

In 2-D can picture sol'ns to Laplace's eq'n by consider a wavy line cut around perimeter of a box. Then stretch rubber membrane over box. The shape of membrane gives sol'ns to Laplace's eq'ns (called Harmonic fcn).



Note ② If you draw a circle of radius R around the pt. (x, y), the average of V around the circle is equal to V at pt. (x, y).

$$\frac{1}{2\pi R} \oint_{\substack{\text{circle of} \\ \text{radius } R}} V(x,y)\, d\ell$$

i.e. V(x,y) is equal to average of V around circumference of circle.

3.1.4 Laplace's eq'n in 3-D

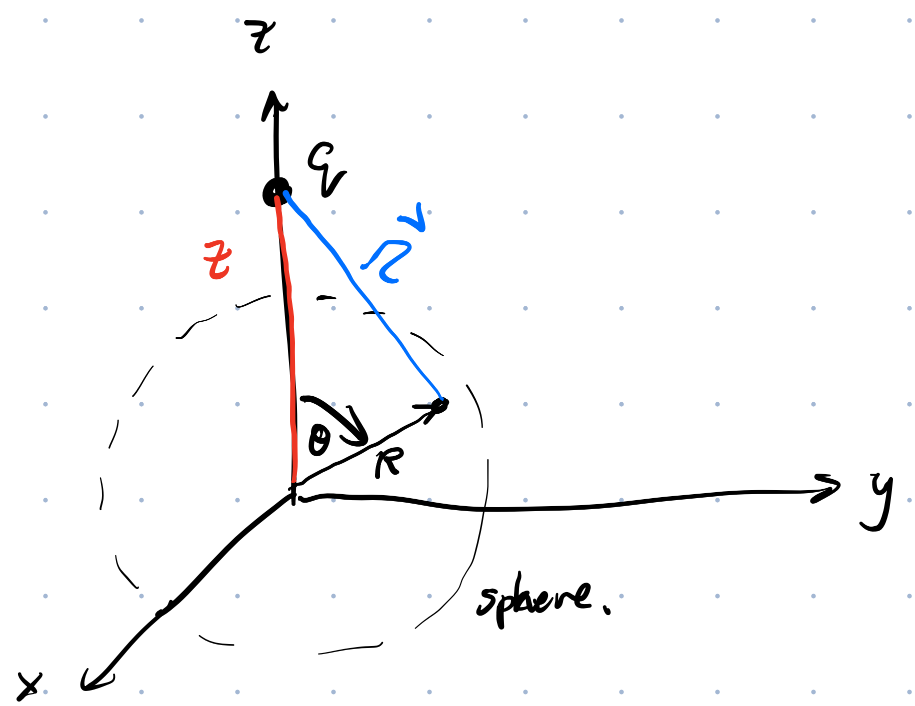
Note ①: V(x,y,z) sol'ns have no extrema.

Note②: Value of V @ (x, y, z) is equal to average of V over a sphere of radius R with pt. (x, y, z) @ centre of sphere.

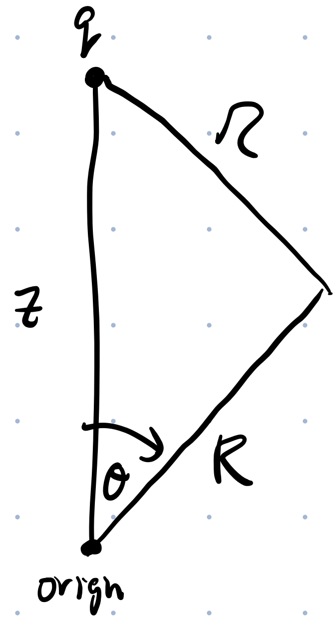
$$V(x,y,z) = \frac{1}{4\pi R^2} \int_{\substack{\text{sphere of} \\ \text{radius } R}} V \, da$$

Proof of ② Consider a pt. charge q on z axis. Find V at (x, y, z) = (0, 0, 0).

Expect $V = \frac{1}{4\pi\varepsilon_0} \frac{q}{z}$



Inside sphere $\rho = 0 \therefore \nabla^2 V = 0$



Know $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathscr{r}}$

Law of cosines

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$$

$$V_{\text{sphere}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}}$$

$$V = \frac{1}{4\pi \cancel{R^2}}\frac{q}{4\pi\varepsilon_0}\int \frac{1}{\sqrt{z^2+R^2-2zR\cos\theta}}\,\cancel{R^2}\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\int_{\theta=0}^{\pi}\frac{\sin\theta\, d\theta}{\sqrt{z^2+R^2-2zR\cos\theta}}$$

$$u = z^2 + R^2 - 2zR\cos\theta$$

$$du = 2zR\sin\theta\, d\theta \therefore \sin\theta\, d\theta = \frac{du}{2zR}$$

when $\theta = 0$, $u = z^2 + R^2 - 2zR = (z-R)^2$

$\theta = \pi$ $\quad u = z^2 + R^2 + 2zR = (z+R)^2$

$$\frac{1}{4zR}\frac{q}{4\pi\varepsilon_0}\int_{u=(z-R)^2}^{(z+R)^2} u^{-1/2}\,du$$

$$= \frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}\, u^{1/2}\Big|_{(z-R)^2}^{(z+R)^2} = \frac{1}{2zR}\frac{q}{4\pi\varepsilon_0}\left[(\cancel{z}+R)-(\cancel{z}-R)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 z} \checkmark$$

Pt. charge a dist. z away. ✓

We can treat any dist'n of charge as a collection of pt. charges that obey the superposition principle. ∴ This proof generalizes to any ρ.

3.1.5 Uniqueness Theorems

First Uniqueness Theorem : The sol'n to $\nabla^2 V = 0$ in a volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary of surrounding surface S.

Proof: Suppose there are two sol'ns, s.t.

$$\nabla^2 V_1 = 0 \quad \& \quad \nabla^2 V_2 = 0$$

with both V_1 & V_2 satisfying boundary conditions.

Consider $V_3 = V_1 - V_2$ } Implies V_3 is zero everywhere on the boundary.

$$\nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \underbrace{\nabla^2 V_1}_{0} - \underbrace{\nabla^2 V_2}_{0} = 0$$

$\therefore$ V_3 satisfies Laplace's eq'n.

Since V_3 has min & max values of zero at boundary, property ① requires that $V_3 = 0$ everywhere.

$$V_3 = V_1 - V_2 = 0 \Rightarrow \boxed{V_1 = V_2}$$

If $\rho \neq 0$ in volume $\mathcal{V}$:

$$\nabla^2 V_1 = -\rho/\varepsilon_0 \qquad \nabla^2 V_2 = -\rho/\varepsilon_0$$

$\therefore$ If $V_3 = V_1 - V_2$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

$\vdots$

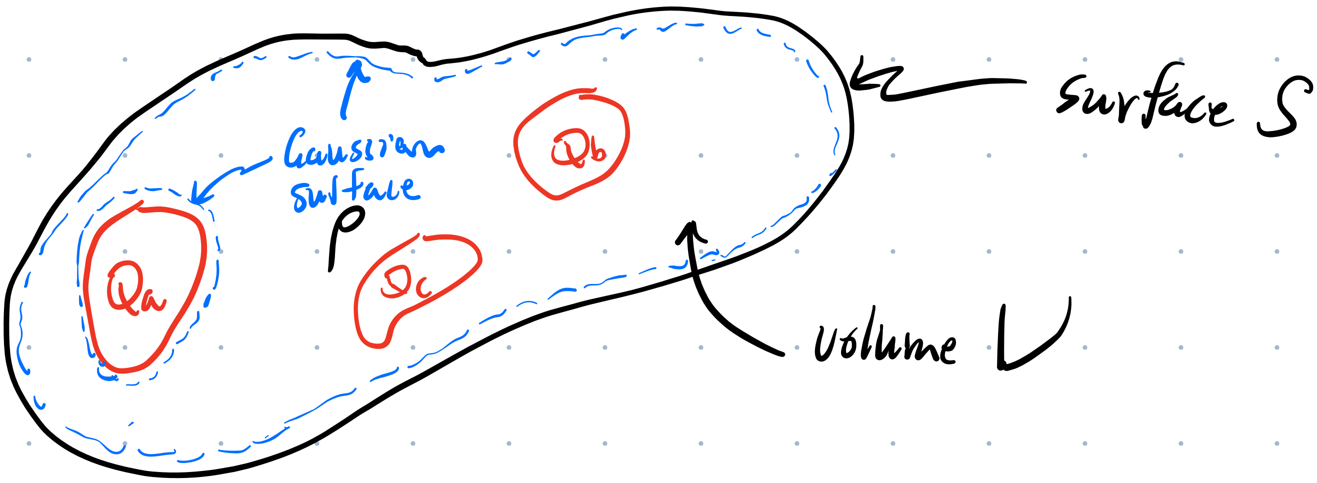
again find $V_1 = V_2$.

Corollary to first uniqueness theorem:

The potential V in volume $\mathcal{V}$ is uniquely determined if charge density ρ in $\mathcal{V}$ and the value of V on all boundaries are specified.

3.1.6 Conductors & 2nd Uniqueness Theorem:

Region of space V bounded by a surface S.



Inside the volume:

- ρ is known
- there are conductors (red) w/ charge Q_i on their surface.

The volume V is either unbounded (i.e. infinite) or surrounded by a conducting surface S.

$\Rightarrow$ In either case, V is constant on bounding surface.

Second Uniqueness Theorem.

In a volume V containing conductors and a specified charge density ρ, $\vec{E}$ is uniquely determined if the total charge on each conductor is given.

Proof: Suppose $\vec{E}_1$ & $\vec{E}_2$ are possible.

In between conductors, each $\vec{E}$ must satisfy Gauss's law in differential form:

$$\vec{\nabla}\cdot\vec{E}_1 = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{E}_2 = \rho/\varepsilon_0$$

They also must satisfy integral form around each conductor.

$$\oint_{\substack{\text{conductor}\\ i}} \vec{E}_1\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} \qquad \oint \vec{E}_2\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0}$$

For a Gaussian surface just inside surrounding boundary

$$\oint_{\text{outer}} \vec{E}_1 \cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0} \qquad \oint_{\text{outer}} \vec{E}_2 \cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0}$$

Like before, examine $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\boxed{\therefore \vec{\nabla} \cdot \vec{E}_3 = 0} \qquad \boxed{\oint \vec{E}_3 \cdot d\vec{a} = 0 \quad \forall \text{ boundaries}}$$

*

Next consider:

$$\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 \underbrace{(\vec{\nabla} \cdot \vec{E}_3)}_{0} + \vec{E}_3 \cdot \underbrace{(\vec{\nabla} V_3)}_{-\vec{E}_3}$$

Product Rule (5)

$$\vec{\nabla} \cdot (V_3 \vec{E}_3) = -E_3^2$$

$$\therefore \int_V \vec{\nabla} \cdot (V_3 \vec{E}_3)\, d\tau = -\int_V E_3^2\, d\tau$$

$\Downarrow$ divergence Theorem

$$\therefore \oint_{\text{all surfaces}} V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V E_3^2\, d\tau$$

All surfaces are next to conductors where V is const. $\therefore V_3 = \text{const.}$

$$V_3 \underbrace{\oint \vec{E}_3 \cdot d\vec{a}}_{\substack{=0 \\ \text{by } (*)}} = -\int_V E_3^2 \, d\tau \Rightarrow E_3 = 0.$$

but since $\vec{E}_3 = \vec{E}_1 - \vec{E}_2 = 0$

$\Rightarrow \vec{E}_1 = \vec{E}_2$