PHYS 301 Oct. 2, 2024

- Assign. #2 on course website.

Last Time: Work & Energy

$$\Delta V = \frac{\Delta U}{Q} \Rightarrow \text{in general} \quad \boxed{V(\vec{r}) = \frac{U(\vec{r})}{Q}}$$

Today: What is the energy of a collection of pt. charges?

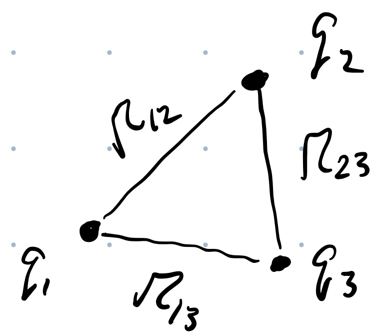
① Move first charge q_1 from ∞ to its final position $\vec{r}_1$.

Work is zero b/c no electric field. $W_1 = 0$.

② Move second charge q_2 into pos. @ $\vec{r}_2$.

Distance between q_1 & q_2 is $\mathscr{r}_{12}$

$$W_2 = \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{q_1}{\mathscr{r}_{12}}}_{\text{potential due to } q_1} q_2$$



③ q_3 from ∞ to $\vec{r}_3$

$$W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right) q_3$$

$$\vdots$$

$$W = W_1 + W_2 + W_3 + \dots$$

$$W = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \dots \frac{q_2 q_3}{\mathscr{r}_{23}} + \dots\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j>i}^{N}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

Excludes $\dfrac{q_2 q_1}{\mathscr{r}_{21}}$ b/c identical to $\dfrac{q_1 q_2}{\mathscr{r}_{12}}$

Instead, can intentionally double count pairs of charges and then divide by 2.

$$\boxed{W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j\neq i}\frac{q_i q_j}{\mathscr{r}_{ij}}}$$

$$= \frac{1}{2}\sum_{i=1}^{N} q_i \underbrace{\left(\sum_{j\neq i} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{\eta_{ij}}\right)}$$

potential at position of q_i due to all other charges.

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)}$$

Energy required to assemble collection of pt. charges.

For a continuous charge dist'n

$$\sum q_i \rightarrow \int \rho \, d\tau$$

$$\boxed{W = \frac{1}{2}\int \rho V \, d\tau}$$

Integrate over volume where $\rho \neq 0$. Can also extend integral to all space if desired since $\rho = 0$ outside charge dist'n.

Next, attempt to re-express W in terms of $\vec{E}$.

Gauss's Law $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$

$$W = \frac{\varepsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E})V\,d\tau$$ integrate by parts...

$$= \frac{\varepsilon_0}{2}\left[\oint V\vec{E}\cdot d\vec{a} - \int \vec{E}\cdot\underbrace{\vec{\nabla}V}_{-\vec{E}}\,d\tau\right]$$

$$W = \frac{\varepsilon_0}{2}\left[\oint_S V\vec{E}\cdot d\vec{a} + \int_V E^2\,d\tau\right]$$

As above, we can select a volume that covers all space. $\therefore$ our surface is out @ ∞ where $V=0$ ($\vec{E}=0$).

$$\boxed{\therefore \quad W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,d\tau}$$

Eg. Find the energy of a uniformly-charge spherical shell of radius R.

Start w/ $W = \frac{1}{2}\int \sigma V \, da$ for a surface dist'n of charge.

At the surface, $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ ← charge of sphere.

$$\sigma = \frac{q}{4\pi R^2}$$

$$da = R^2 \sin\theta \, d\theta \, d\phi$$

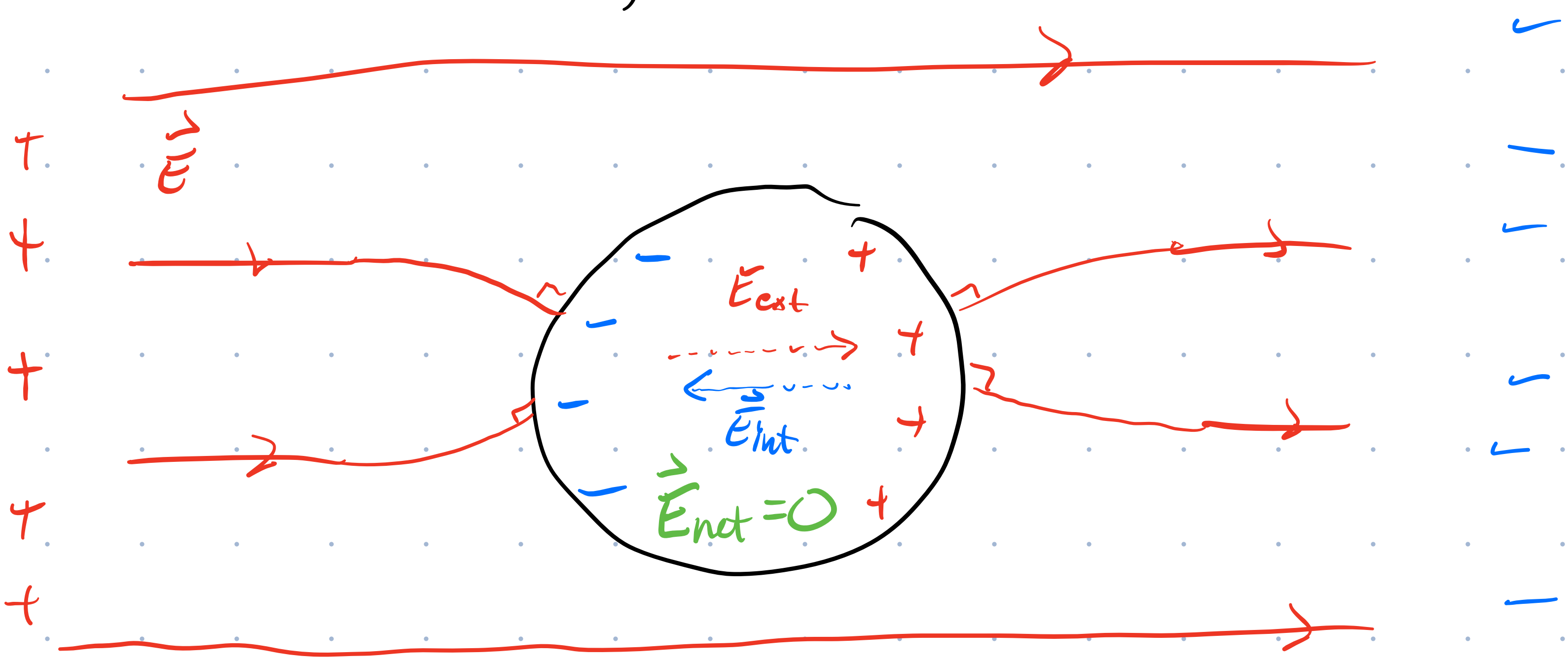
$$W = \frac{1}{2}\int \frac{q}{4\pi R^2}\frac{1}{4\pi\varepsilon_0}\frac{q}{R} \, da$$

$$= \frac{1}{2}\frac{q}{\cancel{4\pi R^2}}\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\underbrace{\int da}_{\cancel{4\pi R^2}} = \boxed{\frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}}$$

2.5 Conductors & Electrostatics.

In a conductor, some e^- in valence shell of atoms leave their host and can move freely throughout the material.

(i) When we place a conductor in an external electric field, the external field drives



the migration of charge in the conductor. Eventually, the internal field $\vec{E}$ due to rearrangement of charge complete cancels the external field.

$\vec{E} = 0$ inside conductors.

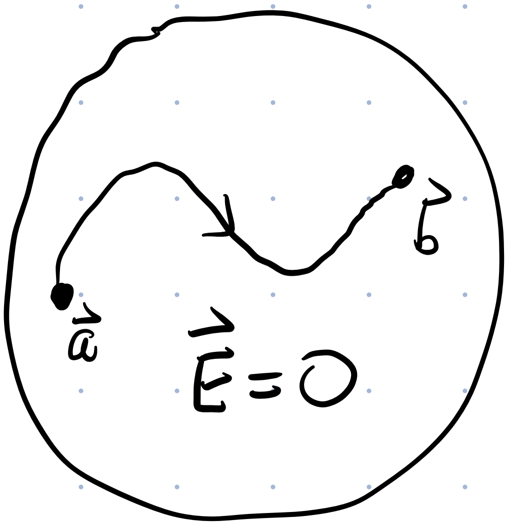
(ii) If $\vec{E}=0$, then by Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

the charge density $\boxed{\rho = 0.}$

Any non-zero net charge on a conductor must reside on a surface.

(iii) Know $\Delta V = -\int_{\vec{a}}^{\vec{b}} \underbrace{\vec{E}}_{0}\cdot d\vec{l}$

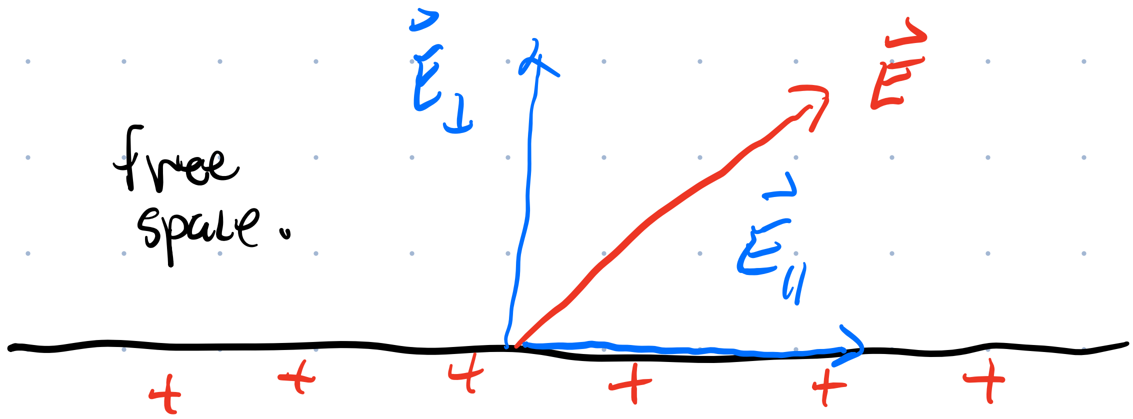


$$\therefore\ V(\vec{b}) - V(\vec{a}) = 0$$

$$V(\vec{b}) = V(\vec{a})$$

Entire conductor is at a constant potential.

(iv) $\vec{E} \perp$ to conductor surface.



conductor $\vec{E}=0$

A charged conductor in an external $\vec{E}$-field.

If this situation ~~=~~ with $\vec{E}_\parallel \neq 0$ was allowed, it would extert a force on charges at surface.

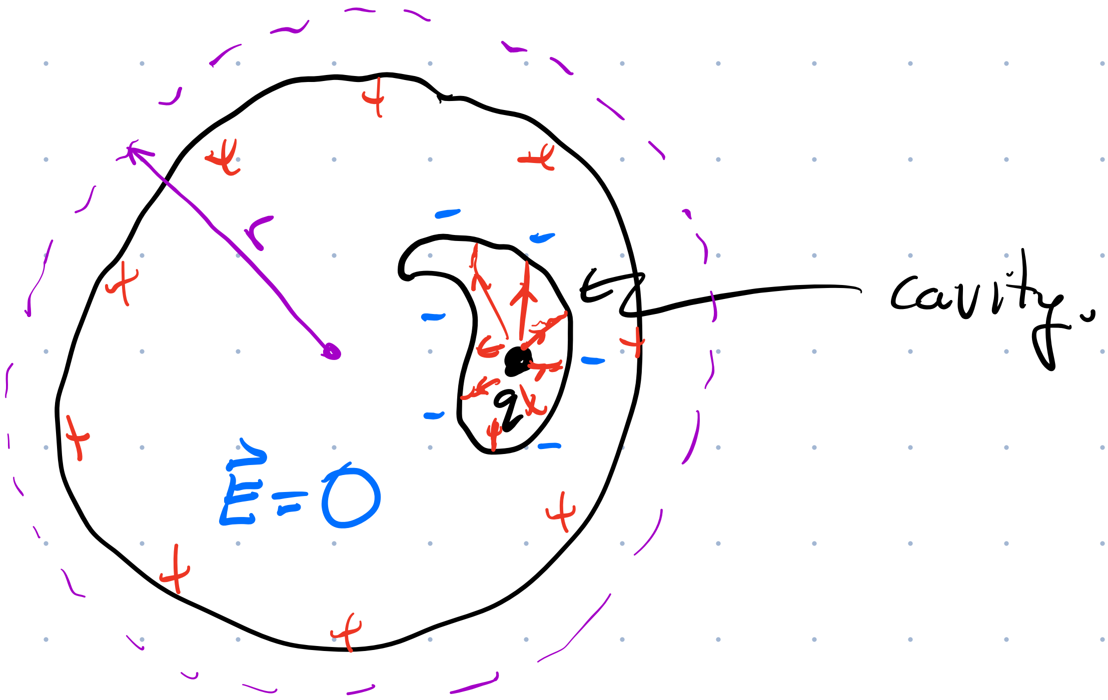
$\Rightarrow$ flow of charge along conductor surface.

$\Rightarrow$ Violates assumption of electrostatics.

$\therefore$ External $\vec{E}$-fields intersect conductors $\perp$ to their surface.

Conductors w/ Cavities

Imagine a neutral spherical conductor w/ a cavity of arbitrary shape and position.



Put a pt. charge q inside cavity at arbitrary position.

Electrons from conductor ~~~~ migrate to cavity surface and cancel its $\vec{E}$ inside bulk of the conductor.

This migration leaves behind excess pos. charge on outer surface.

Because the $\vec{E}=0$ inside conductor, dist'n of charge at outer surface is determined by coulomb repulsion of those charges.

→ i.e. outer surface charge is distributed uniformly.

We can use the intergal form of Gauss's law to find $\vec{E}$ outside the conductor.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$$

$$Q_{encl} = \underbrace{q_{outer} + q_{inner}}_{\text{surface charges}} + q$$

$$q_{outer} = -q_{inner} \Rightarrow q_{outer} + q_{inner} = 0$$

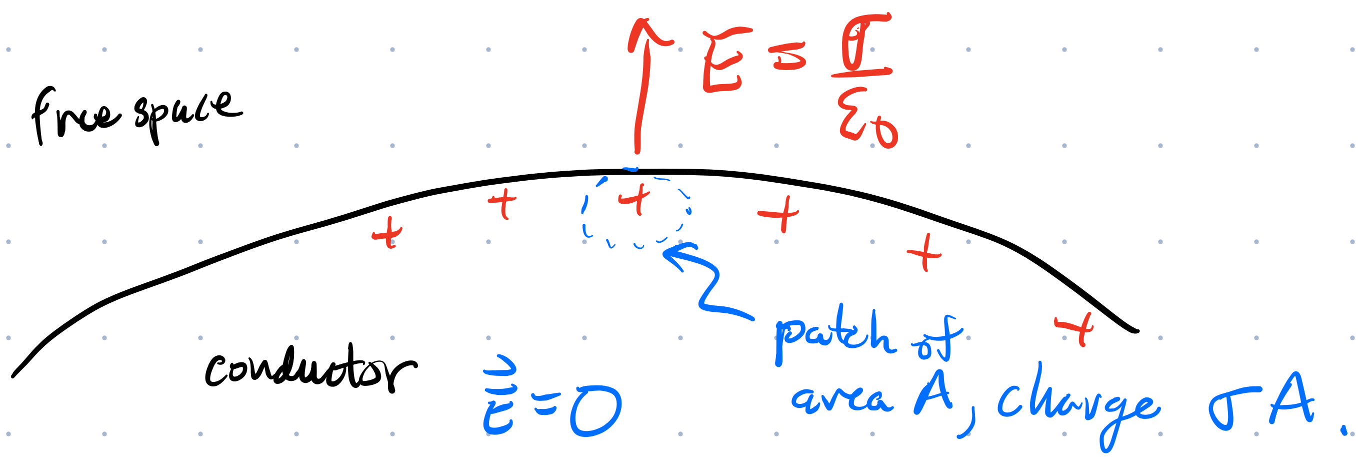
$Q_{encl} = q$

$$\oint \vec{E}\cdot d\vec{a} = \oint E\,da = E\oint da = E4\pi r^2$$

$$E4\pi r^2 = \frac{q}{\varepsilon_0} \implies \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$$

Outside the conductor, it looks as if we have a single pt. charge q at conductor's centre.

2.5.3 Surface Charge & Electrostatic Pressure.



Conductor w/ surface charge density σ.

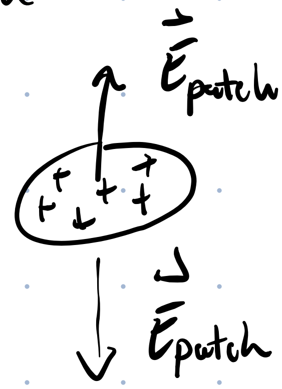
Boundary conditions

$$E^{\perp}_{above} - \underbrace{E^{\perp}_{below}}_{0} = \frac{\sigma}{\varepsilon_0}$$

$$E^{\perp}_{above} = \frac{\sigma}{\varepsilon_0}$$

What is force on a patch of area on conductor due to $\vec{E}$?

Force on patch is due to $\vec{E}$ from all other charge except for $q_{patch} = \sigma A$.



At location of patch:

from all other charges

Above: $\vec{E}_{above} = \vec{E}_{other} + \vec{E}_{patch}$

Below: $\vec{E}_{below} = \vec{E}_{other} - \vec{E}_{patch}$

$$\therefore \frac{1}{2}\left(\vec{E}_{above} + \vec{E}_{below}\right) = \vec{E}_{other}$$

$$|\vec{E}_{other}| = \frac{1}{2}\left(\frac{\sigma}{\varepsilon_0} + 0\right) = \frac{\sigma}{2\varepsilon_0}$$

Force on our patch of area A

$$F_{patch} = E_{other}\, q_{patch}$$

$$= \frac{\sigma}{2\varepsilon_0}\sigma A = \frac{\sigma^2}{2\varepsilon_0} A$$

Electrostatic Pressure: $P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0}$

$$= \frac{1}{2}\varepsilon_0 \underbrace{\left(\frac{\sigma}{\varepsilon_0}\right)^2}$$

E, the total electric field.

$$\boxed{P = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2}$$