PHYS 301 Sept. 25, 2024

Last Time:

Summary for Electrostatics

Gauss's Law $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \Longleftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$$\oint \vec{E} \cdot d\vec{l} = 0 \Longleftrightarrow \vec{\nabla} \times \vec{E} = 0$$

Today: 2.3 The Electric Potential

fcn of ending pt. $\vec{r}$.

Consider $$\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l} = -V(\vec{r})$$

some arbitrary ref. pt. $\rightarrow O$

neg. is by convention

It must be the case that

$$V(\vec{a}) = -\int_{O}^{\vec{a}} \vec{E} \cdot d\vec{l} \qquad V(\vec{b}) = -\int_{O}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

$$\therefore \Delta V = V(\vec{b}) - V(\vec{a}) = -\int_{O}^{\vec{b}} \vec{E}\cdot d\vec{l} - \int_{\vec{a}}^{O} \vec{E}\cdot d\vec{l}$$

$$\boxed{V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}}$$

change in electric potential when we move through $\vec{E}$ from $\vec{a}$ to $\vec{b}$.

must be equal.

From the fundamental theorem of calc for gradients, we know

$$\boxed{V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \vec{\nabla} V \cdot d\vec{l}}$$

$$\Rightarrow \boxed{\therefore \vec{E} = -\vec{\nabla} V}$$

Electric field expressed in terms of the gradient of a scalar.

If we can find V due to a charge dist'n, can deduce $\vec{E}$ from $-\vec{\nabla} V$.

Poisson's Eq'n & Laplace's Eq'n

If $\vec{E} = -\vec{\nabla} V$

then:

① $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$$\Downarrow$$

$$\therefore -\vec{\nabla} \cdot (\vec{\nabla} V) = \frac{\rho}{\varepsilon_0}$$

$$\Downarrow$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$ Poisson's Eq'n.

In a region of space where $\rho = 0$ (charge free region)

$$\boxed{\nabla^2 V = 0}$$ Laplace's Eq'n.

The potential V of a pt. charge.

Know that for a pt. charge $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

$$V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

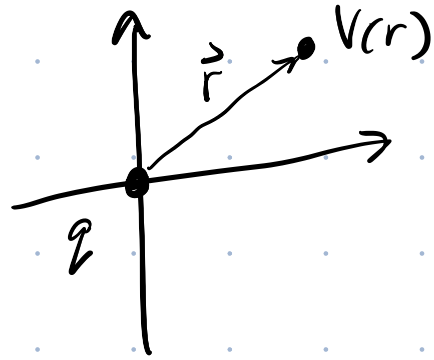
O ← reference pt. is arbitrary. Convenient to define $V(\infty) = 0$ for pt. charges & other finite charge dist'ns.

$$\therefore \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(r')^2} dr'$$

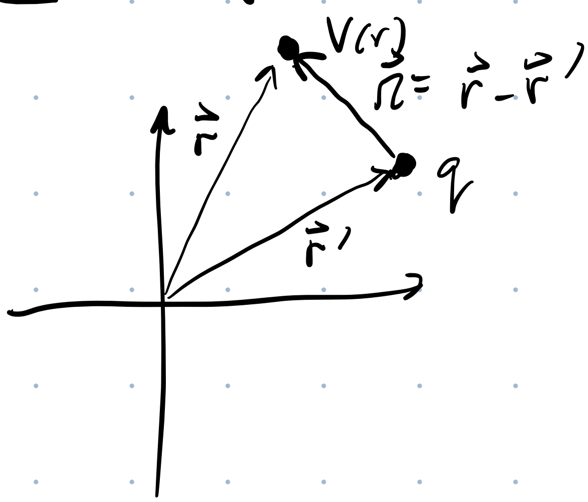
$$V(\vec{r}) = -\frac{q}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{dr'}{(r')^2} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r'} \Bigg|_{\infty}^{r}$$

$$= \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{r} - \cancelto{0}{\frac{1}{\infty}} \right]$$

For a pt. charge $\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}}$ w/ $V = 0$ @ $r \to \infty$.



If q is not @ origin:



$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$$

Electric potential due to many pt. charges:
The electric potential also obeys the superposition principle. ∴

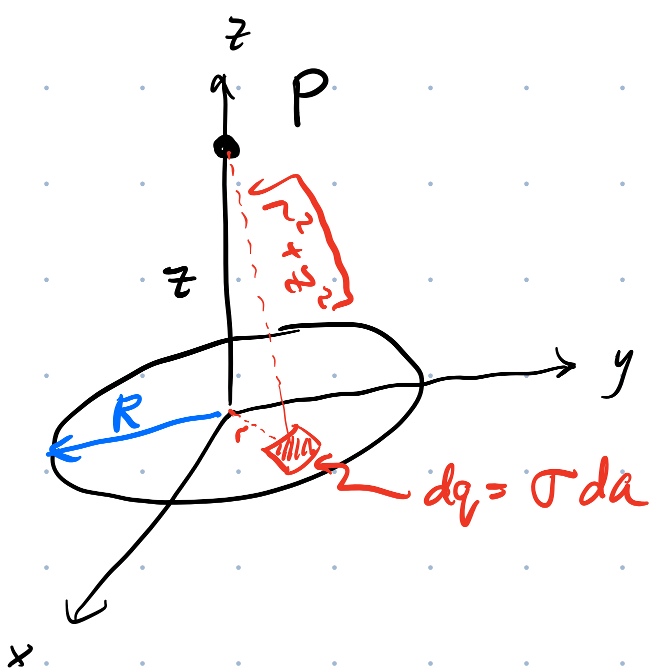
$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i}^{N}\frac{q_i}{\mathscr{r}_i}}$$

For a continuous line of charge $dq = \lambda\, dl$
surface of charge $dq = \sigma\, da$
volume of charge $dq = \rho\, d\tau$

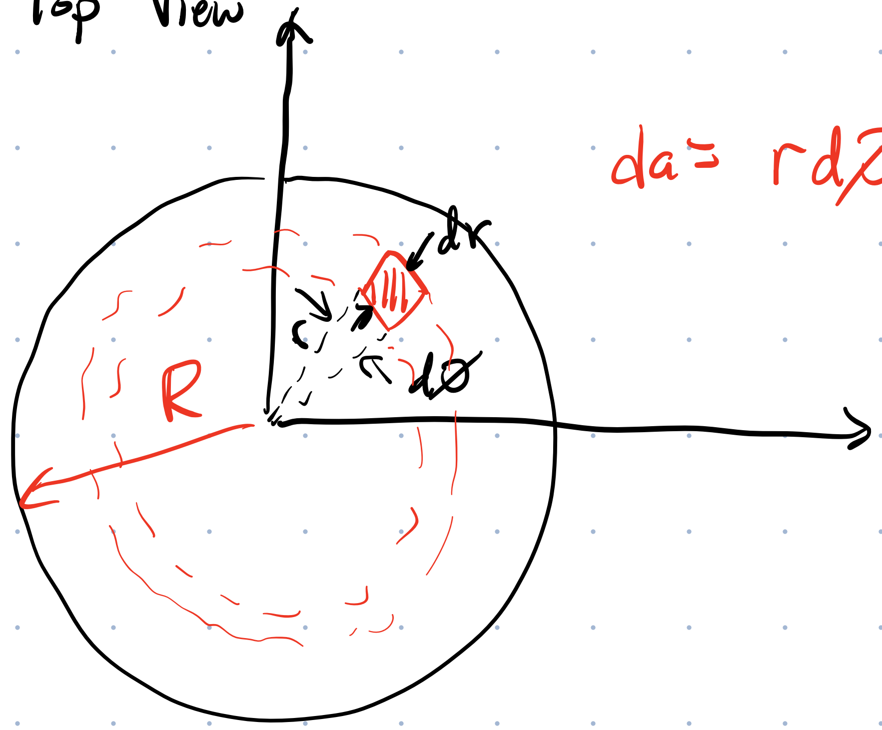
$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}'}\,dl' \\ & \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}\,')}{\mathscr{r}'}\,da' \\ & \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}'}\,d\tau' \end{aligned}}$$

assume $V(\infty) = 0$.

<u>Example</u> Find the potential due to a uniformly-charged disk of radius R for a pt. a height z above centre of disk.



Top View



$da = r\,d\phi\,dr$

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{\mathscr{r}}\,da$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{\sqrt{r^2+z^2}}\,r\,d\phi\,dr$$

$$= \frac{1}{4\pi\varepsilon_0}\underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi}\int_{r=0}^{R}\frac{\sigma\,r\,dr}{\sqrt{r^2+z^2}}$$

$u = r^2+z^2 \qquad du = 2r\,dr$

$r=0, \quad u = z^2$

$r=R, \quad u = R^2+z^2$

$$V = \frac{\sigma}{2\varepsilon_0}\int_{u=z^2}^{R^2+z^2}\frac{1}{2}u^{-1/2}\,du$$

$$= \frac{\sigma}{2\varepsilon_0} u^{1/2} \Big|_{z^2}^{R^2+z^2}$$

$$\boxed{V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - z\right]}$$

Can now find $\vec{E}$ using $\vec{E} = -\vec{\nabla} V$

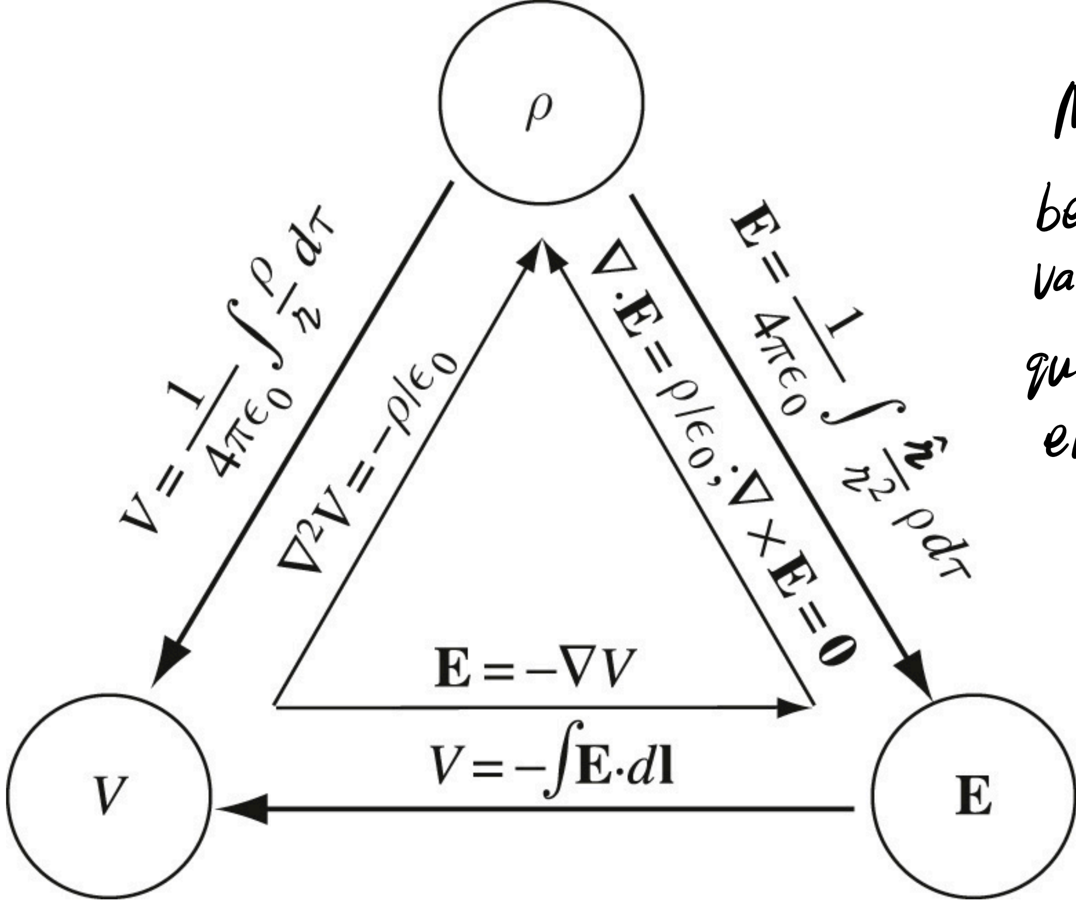
since no ϕ or r dependence in V $\Rightarrow$ $E_z = -\frac{dV}{dz}$

$$E_z = -\frac{d}{dz}\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - z\right]$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]$$

$$\boxed{\therefore \vec{E} = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{z}}$$

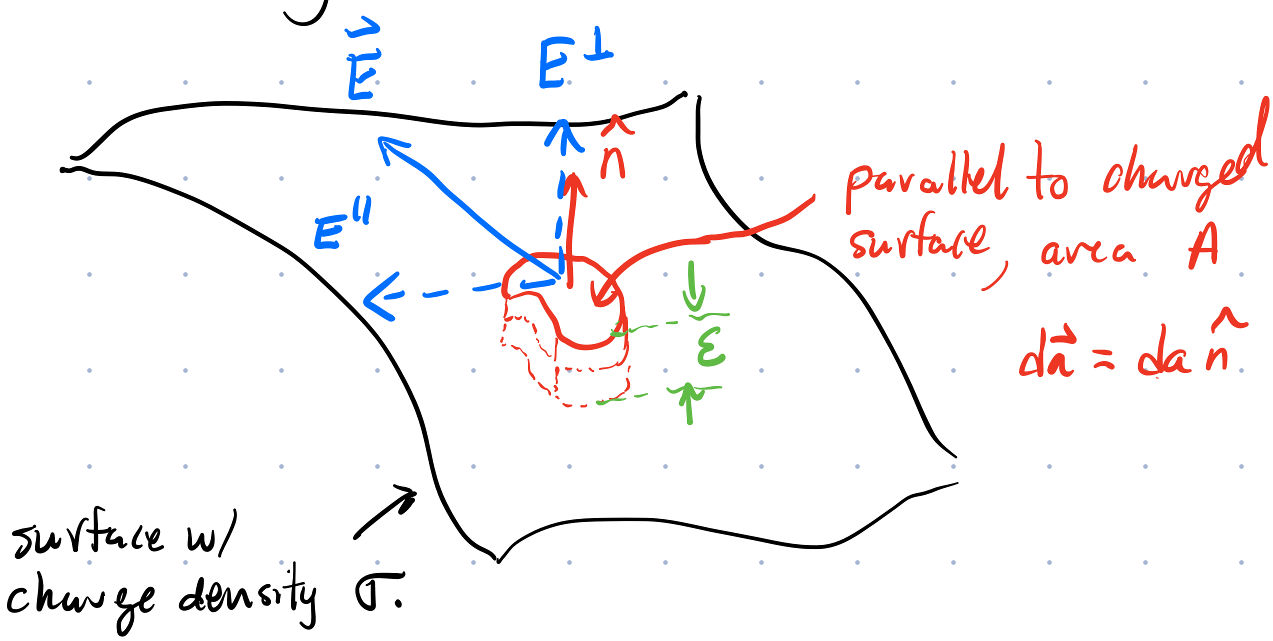
Electric above the centre of a uniformly-charged disk.



Map to go between the various fundamental quantities in electrostatics.

2.3.5 Boundary Conditions

How does $\vec{E}$ change at the boundary of a object carrying surface σ?



Apply Gauss's law to the surface.

$$\oint \vec{E}\cdot d\vec{a}$$

When we evaluate $\vec{E}\cdot d\vec{a}$, since $d\vec{a}$ is in $\hat{n}$ dir'n, $\perp$ to charged surface, only the $\perp$ component of $\vec{E}$ survives.

$$\vec{E}\cdot d\vec{a} = E^{\perp} da$$

$$\oint_{\text{surface}} \vec{E}\cdot d\vec{a} = \int_{\text{top}} \underbrace{\vec{E}\cdot d\vec{a}}_{E^{\perp}_{top}da} + \int_{\text{btm}} \underbrace{\vec{E}\cdot d\vec{a}}_{-E^{\perp}_{btm}da} + \underbrace{\int_{\text{side}} \vec{E}\cdot d\vec{a}}_{\text{zero in limit } \varepsilon\to 0 \text{ (infinitesimally thin Gaussian surface).}}$$

$$\therefore \int E^{\perp}_{top}\, da - \int E^{\perp}_{btm}\, da = \frac{Q_{encl}}{\varepsilon_0}$$

but $\sigma A = Q_{encl}$

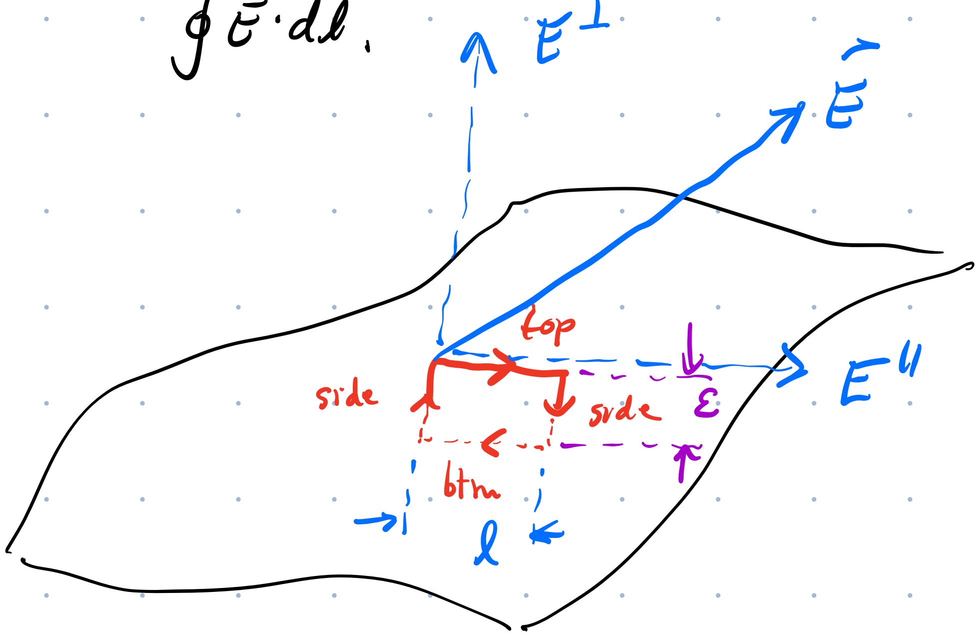
& for sufficiently-small values of A, can assume $E^{\perp}_{top,btm} \approx$ const.

$$E^{\perp}_{top}\cancel{A} - E^{\perp}_{btm}\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

$$\boxed{E^{\perp}_{top} - E^{\perp}_{btm} = \frac{\sigma}{\varepsilon_0}}$$

$\perp$ component of $\vec{E}$ changes abruptly when cross a boundary w/ surface charge density σ.

To see what happens to parallel component of $\vec{E}$, consider $\oint \vec{E}\cdot d\vec{l}$.



top: $\vec{E}\cdot d\vec{l} = E^{\parallel}_{top}\, dl$

btm: $\vec{E}\cdot d\vec{l} = -E^{\parallel}_{btm}\, dl$

sides: $\vec{E}\cdot d\vec{l} \to 0$ in limit $\varepsilon \to 0$

$$\oint \vec{E}\cdot d\vec{l} = E^{\parallel}_{top}\cancel{l} - E^{\parallel}_{btm}\cancel{l} = 0$$

$$\boxed{\therefore\ E^{\parallel}_{top} = E^{\parallel}_{btm}}$$

Parallel component of $\vec{E}$ continuous across a charged boundary/surface.

2.4 Work & Energy

Work-KE Theorem: $W = \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l} = \Delta K$

sub $\vec{F} = Q\vec{E}$

$\Delta K = -\Delta U$ (for conserv. forces, mech. energy is conserved).

$$\int_{\vec{a}}^{\vec{b}} Q\vec{E}\cdot d\vec{l} = -\Delta U$$ divide by $-Q$

$$\underbrace{\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}}_{\cancel{-}\left[V(\vec{b}) - V(\vec{a})\right]} = \cancel{-}\frac{\Delta U}{Q}$$

$$\boxed{\therefore\ \Delta V = \frac{\Delta U}{Q}}$$

relationship between electric potential & the electric P.E.