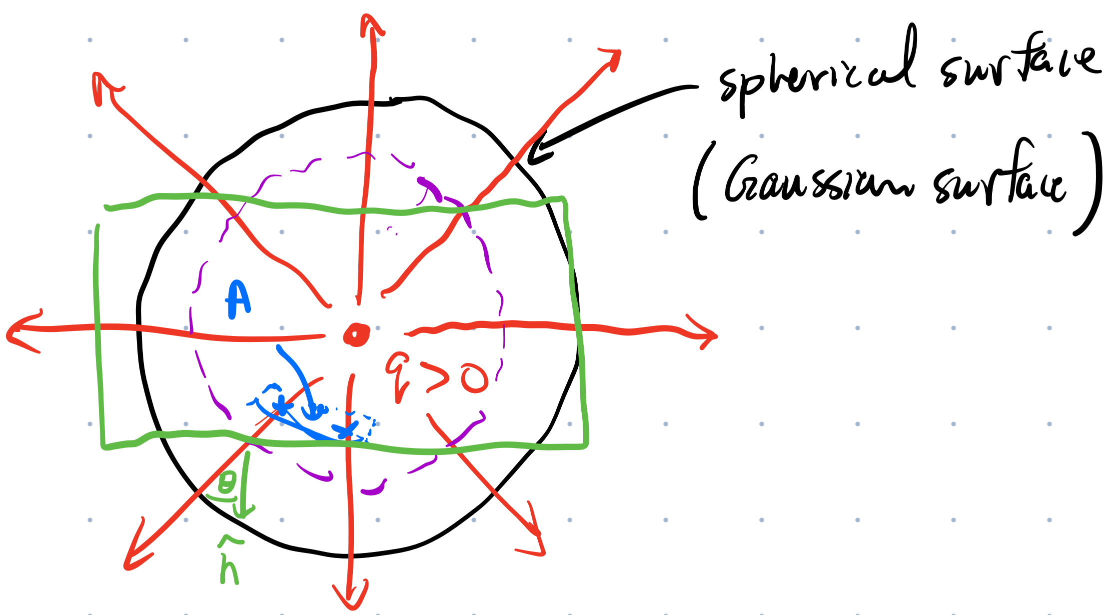
PHYS 301 Sept. 23, 2024

2.2 Gauss's Law

Calc. $\vec{E}$ due to continuous charge dist'n is possible using methods discussed previously. But in some situations those calculations can be cumbersome. Want new methods to find $\vec{E}$.

Start by calc. the flux of $\vec{E}$ through a surface that surrounds a pt. charge. To make this calc. easy, pick a spherical surface w/ pt. charge at centre.



Electric field strength is prop. to density of lines (i.e. $\frac{\#\text{lines}}{\text{area}}$ crossing a surface.)

The flux $\Phi_E = \oint \vec{E} \cdot d\vec{a}$ (like $\vec{E}$-field strength × area
$\frac{\#\text{lines}}{\text{area}} \cdot \text{area} = \#\text{lines}$)

Black & purple sphere's have same Φ_E since same no. of lines cross those surfaces.

Likewise, the rectangular green surface also has the same Φ_E, although actually evaluating $\oint \vec{E} \cdot d\vec{a}$ would be very difficult.

For a sphere of radius R w/ q @ centre

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \Rightarrow \underset{@\text{surface}}{\frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{r}}$$

$$\Phi_E = \oint \vec{E} \cdot d\vec{a}$$

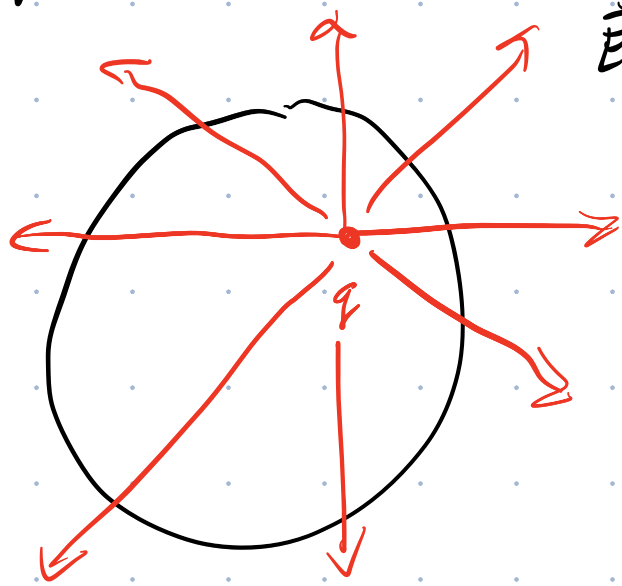
$$d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\vec{E} \cdot d\vec{a} = \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{\cancel{R^2}} \hat{r} \right) \cdot \left(\cancel{R^2} \sin\theta \, d\theta \, d\phi \, \hat{r} \right)$$

$$= \frac{q}{4\pi\varepsilon_0} \sin\theta \, d\theta \, d\phi$$

$$\Phi_E = \frac{q}{4\pi\varepsilon_0} \underbrace{\int_{\substack{\text{sphere} \\ \text{surface}}} \sin\theta \, d\theta \, d\phi}_{4\pi} = \frac{q}{\varepsilon_0}$$

If q is moved off centre, still have same no. of $\vec{E}$-field lines crossing surface & Φ_E is unchanged.

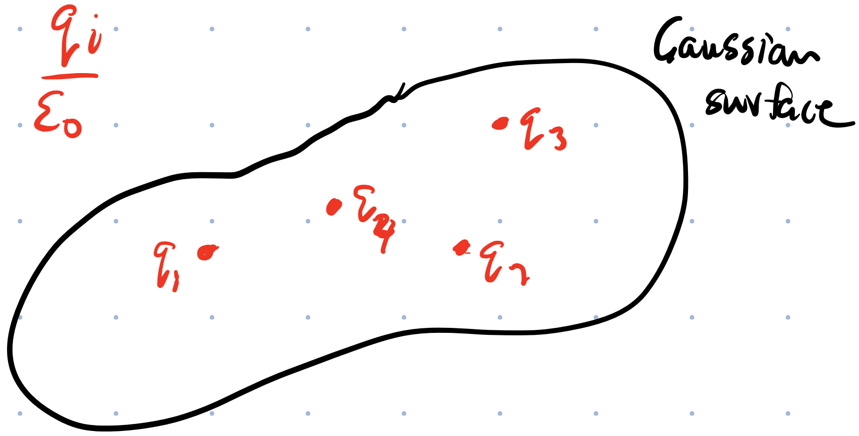


What if there are multiple pt. charges inside our surface.

$$\Phi_E = \oint \left(\sum_{i=1}^{N} \vec{E}_i \right) \cdot d\vec{a}$$

$$= \sum_{i=1}^{N} \oint \vec{E}_i \cdot d\vec{a} = \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i$$

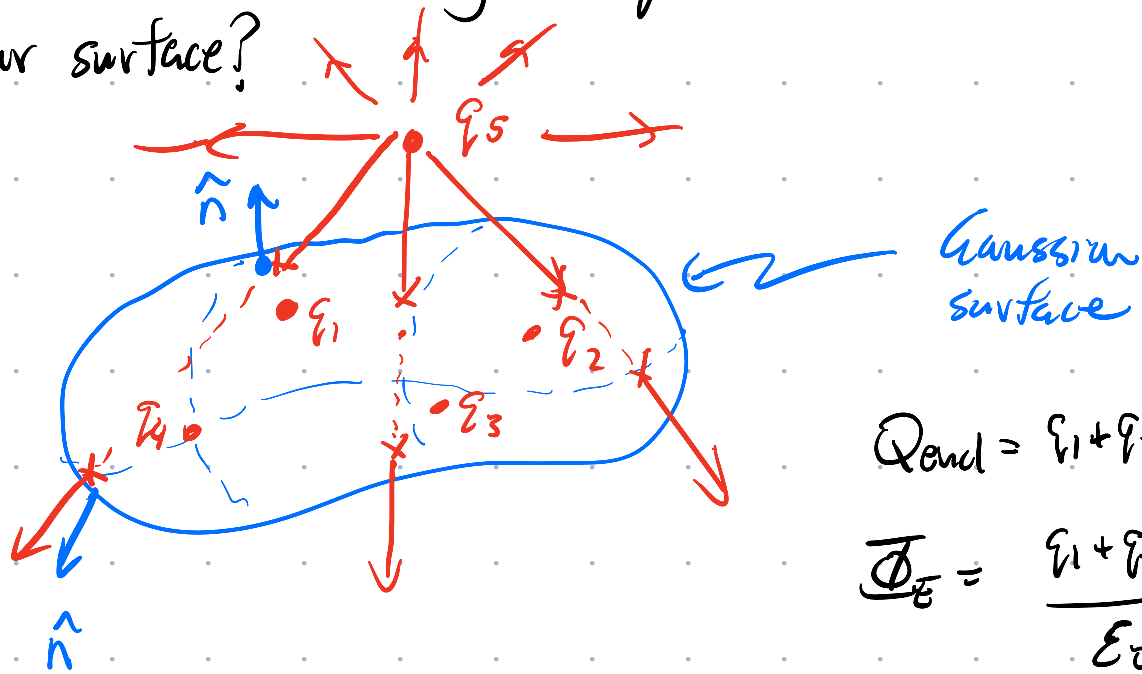
$$\frac{q_i}{\varepsilon_0}$$



$$\sum_{i=1}^{N} q_i = Q_{encl}$$ total charge contained within Gaussian surface.

$$\boxed{\text{Gauss's Law} \quad \Phi_E = \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}}$$

What if have nearby charges that are <u>outside</u> our surface?



$$Q_{encl} = q_1 + q_2 + q_3 + q_4$$

$$\Phi_E = \frac{q_1 + q_2 + q_3 + q_4}{\varepsilon_0}$$

$\oint \vec{E}\cdot d\vec{a}$

$\vec{E}\cdot d\vec{a} < 0$ when $\vec{E}$ & $d\vec{a}$ approx. antiparallel

→ contribute negative flux

$\vec{E}\cdot d\vec{a} > 0$ when $\vec{E}$ & $d\vec{a}$ are approx parallel

→ positive flux.

For charges outside surface, the pos. & neg. contrib. to Φ_E exactly cancel.

Before doing our first example, let's apply divergence theorem to Gauss's law.

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{1}{\varepsilon_0} Q_{encl}$$

$$= \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \qquad \text{integral form}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{differential form.}$$

While Gauss's law is always true, it is only easy to apply when we satisfy the following conditions:

① $\vec{E}$ should $\parallel$ or $\perp$ to $d\vec{a}$ everywhere on surface. $\Rightarrow$ allows us to evaluate the dot product $\vec{E} \cdot d\vec{a}$

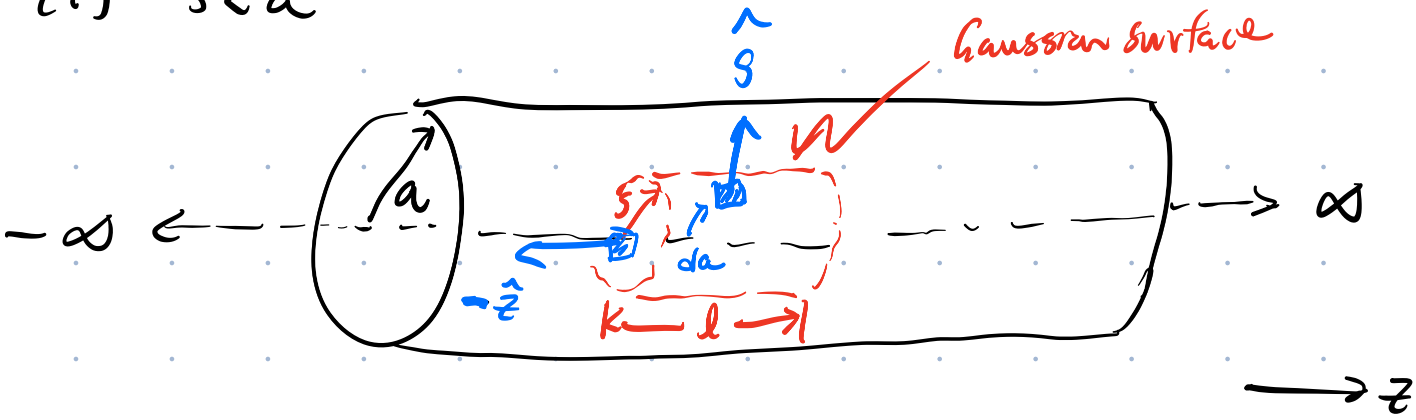
② Wherever $\vec{E} \cdot d\vec{a} \neq 0$, require $|\vec{E}|$ to be zero or constant, so that E can be factored out of the integral.

Eg. Long cylinder carries a charge density

$\rho = ks^2$ $\qquad$ s is radial coord. in cylindrical coords.

If the cylinder is radius a, find $\vec{E}$ at pts w/ $s < a$ (inside) & $s > a$ (outside).

(i) $s<a$



$\vec{E}$ will always be radial. $\therefore \vec{E} = E_s \hat{s}$

Start w/ $\oint \vec{E} \cdot d\vec{a}$

curved surface $\vec{E} \cdot d\vec{a} = (E_s \hat{s}) \cdot (da\, \hat{s}) = E_s\, da$

left end $\vec{E} \cdot d\vec{a} = (E_s \hat{s}) \cdot (-da\, \hat{z}) = 0$

right end $\vec{E} \cdot d\vec{a} = 0$

$$\oint \vec{E} \cdot d\vec{a} = \underbrace{\int_{\substack{\text{left}\\ \text{end}}} \cancel{\vec{E} \cdot d\vec{a}}}_{0} + \underbrace{\int_{\substack{\text{right}\\ \text{end}}} \cancel{\vec{E} \cdot d\vec{a}}}_{0} + \int_{\text{curved}} \vec{E} \cdot d\vec{a}$$

$$= \int_{\text{curved}} E_s\, da$$

Since Gaussian (curved part) surface is always the same dist. s from cylinder axis, E_s is const. & can be take outside the integral.

$$\Phi_E = \oint \vec{E}\cdot d\vec{a} = E_s \int_{\text{curved}} da = E_s \underbrace{a_{\text{curved}}}_{2\pi s l}$$

$$\boxed{\therefore \Phi_E = E_s\, 2\pi s l}$$

This result will be the same inside & outside.
$(s<a)$ $(s>a)$

Next, consider $\frac{Q_{encl}}{\varepsilon_0}$.

$$Q_{encl} = \int_{\substack{\text{Gaussian}\\ \text{surface}}} \rho \, d\tau$$

$d\tau = s' ds' d\phi dz$

$\rho = k(s')^2$

$$= \int_{\substack{\text{Gaussian}\\ \text{surface}}} k(s')^2 s' ds' d\phi \, dz$$

$0 < z < l$

$0 < s' < s$

$0 < \phi < 2\pi$

$$Q_{encl} = k \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{l} dz}_{l} \underbrace{\int_{s'=0}^{s} (s')^3 ds'}_{s^4/4}$$

$$\boxed{Q_{encl} = \frac{\pi \ell k s^4}{2}}$$

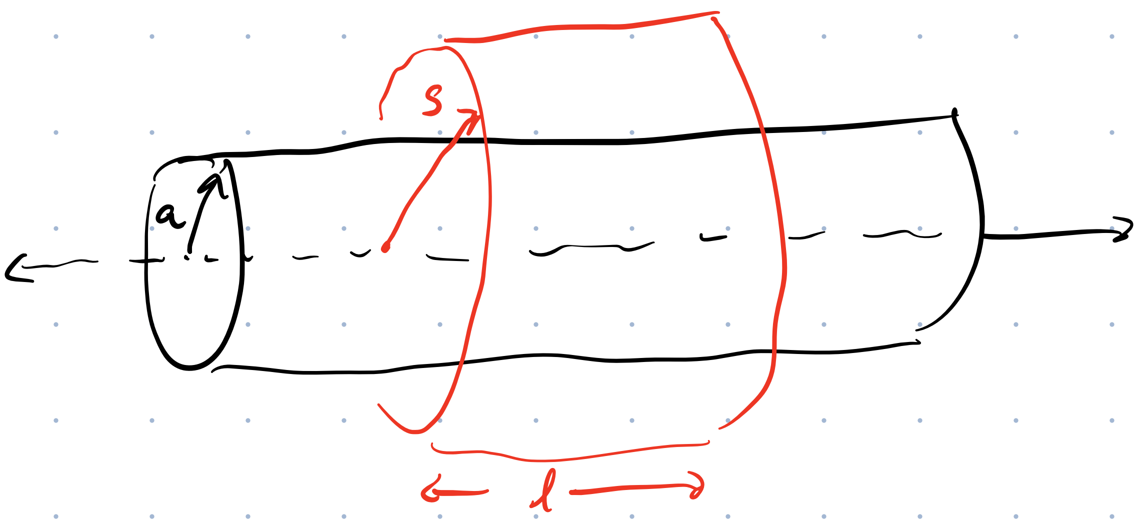
$\therefore \oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$ requires

$$E_s\, 2\cancel{\pi}\cancel{s}\cancel{\ell} = \frac{\cancel{\pi}\cancel{\ell} k s^{\cancel{4}\,3}}{2\varepsilon_0}$$

$$\therefore E_s = \frac{k s^3}{4\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{k s^3}{4\varepsilon_0}\hat{s} \text{ for } 0<s<a \text{ inside cylinder.}}$$

(ii) $s > a$ (outside)

$Q_{encl} = \int \rho \, d\tau$



In this case $\oint \vec{E}\cdot d\vec{a} = E_s\, 2\pi s \ell$ is exactly the same.

$$Q_{encl} = \int_{\substack{\text{charge} \\ \text{cylinder}}} \rho \, d\tau = \int_{s'=0}^{a} \int_{z=0}^{l} \int_{\phi=0}^{2\pi} k(s')^2 \, s' \, ds' \, d\phi \, dz$$

$$= k \, 2\pi l \int_0^a (s')^3 \, ds'$$

$$= \frac{k \pi l a^4}{2}$$

$$\left\{ \text{Think about charge per unit length of cylinder} \quad \lambda = \frac{Q_{encl}}{l} = \frac{k\pi a^4}{2} \right\}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E_s \, 2\cancel{\pi} s \cancel{l} = \frac{k \cancel{\pi} a^4 \cancel{l}}{2 \varepsilon_0}$$

$$E_s = \frac{k a^4}{4 \varepsilon_0 s} \Rightarrow \boxed{\vec{E} = \frac{\boxed{k a^4}}{4 \varepsilon_0 s} \hat{s}}$$

Rewrite $\vec{E}$ in terms of λ.

$$k a^4 = \boxed{\frac{2\lambda}{\pi}}$$

$$\therefore \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\,\hat{s}}$$

Electric field due to a thin wire w/ charge per unit length λ.

2.2.4: The Curl of $\vec{E}$

Consider a pt. charge for which $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

If we select $r_a = r_b$ (closed loop), then

$$\oint \vec{E}\cdot d\vec{l} = 0$$

For a collection of pt. charges $\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots$

$$\oint(\vec{E}_1+\vec{E}_2+\cdots)\cdot d\vec{l} = \underbrace{\oint \vec{E}_1\cdot d\vec{l}}_{0} + \underbrace{\oint \vec{E}_2\cdot d\vec{l}}_{0} + \cdots = 0$$

Apply Stoke's theorem $\oint_P \vec{v}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{a}$

$$0 = \oint \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} \Longrightarrow \vec{\nabla}\times\vec{E} = 0$$

true for electro-statics.