PHYS 301 Sept. 18, 2024

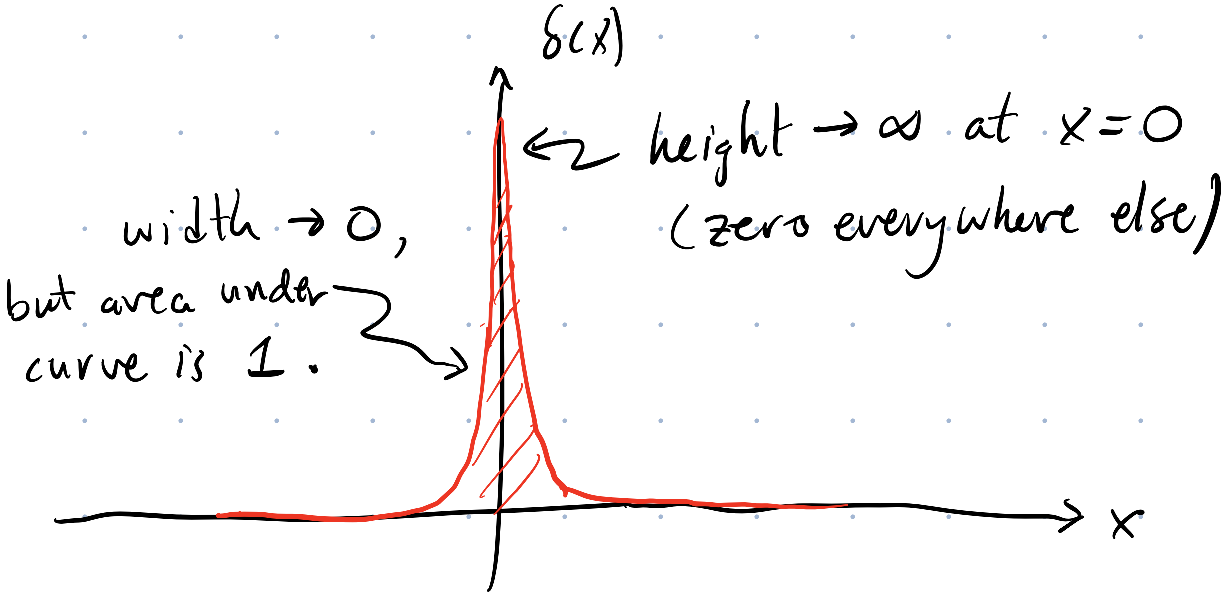
<u>Last Time</u>: The Dirac-Delta fcn $\delta(\vec{r})$:

$$4\pi\,\delta(\vec{r}) \equiv \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

Using the divergence theorem, we showed:

$$\int \delta(\vec{r})\,d\tau = 1$$

In 1-D the Dirac-Delta fcn is visualized as:



<u>Today</u>: Using δ-fcns to evaluate integrals.

Consider the product $\delta(\vec{r})f(\vec{r})$

This product must be equiv. to $\delta(\vec{r})f(\vec{0})$ since $\delta(\vec{r})=0$ everywhere except $\vec{r}=\vec{0}$.

$$\therefore \int \delta(\vec{r})f(\vec{r})\,d\tau = \int \delta(\vec{r})f(\vec{0})\,d\tau$$

no longer a fcn $\vec{r}$. It is a const. that can be taken outside integral.

$$= f(\vec{0})\underbrace{\int \delta(\vec{r})\,d\tau}_{=1}.$$

$$\therefore \boxed{\int \delta(\vec{r})f(\vec{r})\,d\tau = f(\vec{0})}$$

value of fcn @ $\vec{r}=0$.

True provided the volume of integration contains the pt $\vec{r}=0$.

We don't need to place the spike of δ-fcn at origin. Consider:

$$\delta(\vec{r}-\vec{r}\,') = \begin{cases} 0 & \text{when } \vec{r} \neq \vec{r}\,' \\ \infty & \text{when } \vec{r} = \vec{r}\,' \end{cases}$$

spike where $\vec{r}-\vec{r}\,' = \vec{0}$ or $\vec{r} = \vec{r}\,'$

In this case, $\delta(\vec{r}-\vec{r}\,')\, f(\vec{r}) = \delta(\vec{r}-\vec{r}\,')\, f(\vec{r}\,')$

$$\boxed{\begin{aligned} \int \delta(\vec{r}-\vec{r}\,') f(\vec{r})\, d\tau &= \int \delta(\vec{r}-\vec{r}\,') \underbrace{f(\vec{r}\,')}_{\text{indep. of } \vec{r}}\, d\tau \\ &= f(\vec{r}\,') \end{aligned}}$$

If $\displaystyle\int \delta(\vec{r}-\vec{r}\,')\, d\tau = 1$ (as it is),

then $[\delta(\vec{r}-\vec{r}\,')] = \dfrac{1}{\text{length}^3}$

Examples

1-D example. Griffiths 1.44(d)

Evaluate $$I = \int_{-\infty}^{\infty} \ln(x+3)\, \delta(x+2)\, dx$$

δ-fcn selects the pt. $x+2=0 \Rightarrow x=-2$

$$= \ln(x+3)\Big|_{x=-2} = \ln(-2+3) = \ln(1) = 0.$$

3-D Example Griffiths 1.48(a)

Evaluate $$I = \int_{\substack{\text{all}\\\text{space}}} (r^2 + \vec{r}\cdot\vec{a} + a^2)\, \delta(\vec{r}-\vec{a})\, d\tau$$

δ-fcn selects $\vec{r}$ s.t. $\vec{r}-\vec{a}=0$

$\therefore \vec{r}=\vec{a}$

$$I = (r^2 + \vec{r}\cdot\vec{a} + a^2)\Big|_{\vec{r}=\vec{a}}$$

$$= a^2 + \underbrace{\vec{a}\cdot\vec{a}}_{a^2} + a^2 = \underline{\underline{3a^2}}$$

Chapter 2: Electrostatics

Goal is to write down the force on test charge Q due to a collection of source charges q_i or a continuous dist'n of charge ρ.

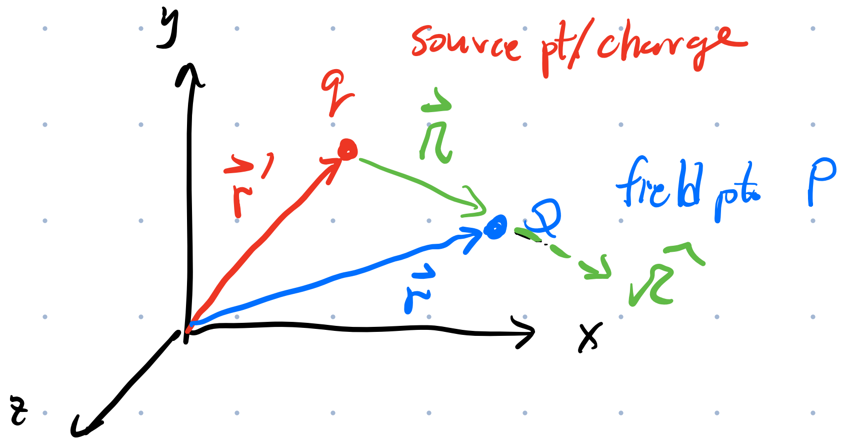
In this chapter we will limit ourselves to the situation in which the source charges are at rest. That is the meaning of electrostatics.

Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$$k_e = \frac{1}{4\pi\varepsilon_0}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$



$\vec{F}$ is the force on pt charge Q due to pt. charge q.

If have many source charges q_i where $i = 1 .. N$, then the net force on Q is given by

$$\begin{aligned} \vec{F} &= \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_N \\ &= \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i \end{aligned}$$

Superposition principle.

Electric Field

Can express the net force of Q in terms of an electric field $\vec{E}$ due to collection of source charges q_i:

$$\vec{F} = Q\vec{E}$$

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

vector sum of electric fields due to each individ source charge.

Continuous Charge Distributions

Will often consider charged objects that will be characterized by a charge density.

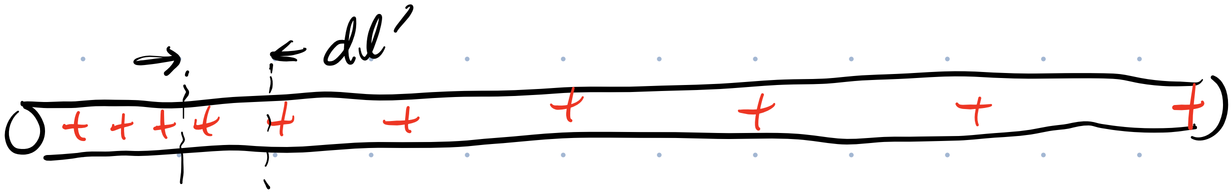
Can have:

linear charge density such as a charge wire:

charge density is $\lambda(\vec{r}\,')$

$$\left[\lambda(\vec{r}\,')\right] = \frac{C}{m}$$

Imagine dividing charged wire into short segments of length dl':



$$\Delta q_i = \lambda(\vec{r}\,')\, dl'$$

We treat each segment as a pt. Add contributions of each segment to find net $\vec{E}$ at some pt. in space.

Use ⊛

$$\vec{E} \approx \frac{1}{4\pi\varepsilon_0} \sum_i \frac{\Delta q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

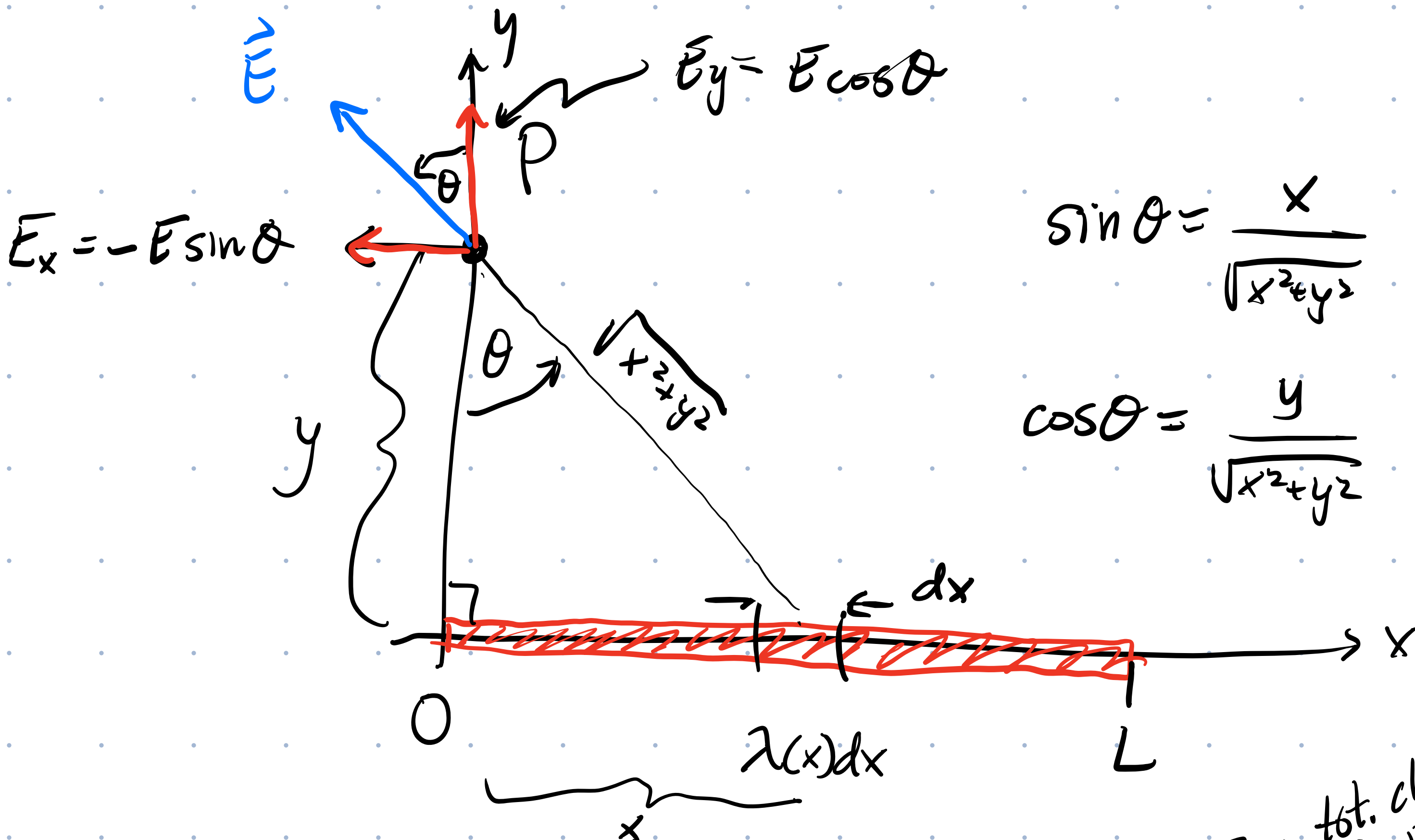
$$= \frac{1}{4\pi\varepsilon_0} \sum_i \frac{\lambda(\vec{r}\,')\, dl'}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

In the limit $dl' \to 0$, sum becomes an integral

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\, dl'\, \hat{\mathscr{r}}}$$

$\vec{E}$ due to a linear dist'n of charge.

Eg. Find $\vec{E}$ @ P due to uniformly charge wire that extends from $x=0$ to L.



$$\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Since wire is uniformly charged, $\lambda = \frac{q}{L}$ ⟸ tot. charge of wire

x-component:

$$E_x = -\frac{1}{4\pi\varepsilon_0}\int_{x=0}^{L} \frac{\lambda}{x^2+y^2}\,\frac{x}{\sqrt{x^2+y^2}}\,dx$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\int_{x=0}^{L} \frac{x}{(x^2+y^2)^{3/2}}\,dx$$

Substitution $u = x^2 + y^2$

$du = 2x\,dx \Rightarrow x\,dx = du/2$

$\vdots$

find

$$\boxed{E_x = -\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{y} - \frac{1}{\sqrt{L^2+y^2}}\right]}$$

y-component

$$E_y = \frac{1}{4\pi\varepsilon_0}\int_{x=0}^{L} \frac{\lambda}{(x^2+y^2)} \frac{y}{\sqrt{x^2+y^2}}\,dx$$

$$= \frac{\lambda y}{4\pi\varepsilon_0}\int_{x=0}^{L} \frac{dx}{(x^2+y^2)^{3/2}}$$

Try sub $x = y \tan\theta$

$$dx = \frac{y}{\cos^2\theta}\,d\theta$$

$\vdots$

find

$$E_y = \frac{\lambda L}{4\pi\varepsilon_0 y\sqrt{L^2+y^2}}$$ recall $\lambda L = q$

$$\boxed{\therefore E_y = \frac{q}{4\pi\varepsilon_0 y\sqrt{L^2+y^2}}}$$

$$\vec{E} = E_x\,\hat{x} + E_y\,\hat{y}$$

Check the limit $y \to \infty$. In this, expect wire to look like pt. charge s.t. $E_x = 0$

$$E_y = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}$$

x-component $y \to \infty$

$$\frac{1}{y} \to 0 \qquad \frac{1}{\sqrt{L^2+y^2}} \to \frac{1}{\sqrt{y^2}} \to \frac{1}{y} \to 0$$

$\therefore E_x = 0$ ✓

y-component $y \to \infty$

$$\frac{1}{4\pi\varepsilon_0} \frac{q}{y\sqrt{L^2+y^2}} \to \frac{1}{4\pi\varepsilon_0} \frac{q}{y\sqrt{y^2}}$$

$$\to \frac{1}{4\pi\varepsilon_0} \frac{q}{y^2} \quad ✓$$

Likewise, we can write expressions for surface charge densities:

σ charge per unit area

$$[\sigma] = \frac{C}{m^2}$$

& volume charge densities:

ρ charge per unit volume

$$[\rho] = \frac{C}{m^3}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')\,da'}{\mathcal{R}^2}\,\hat{\mathcal{R}} \quad \text{surface}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho\frac{(\vec{r}')\,d\tau'}{\mathcal{R}^2}\,\hat{\mathcal{R}} \quad \text{volume.}$$