PHYS 301 Sept. 16, 2024

Last Time

Stoke's Theorem:

$$\int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a} = \oint_P \vec{V} \cdot d\vec{l}$$

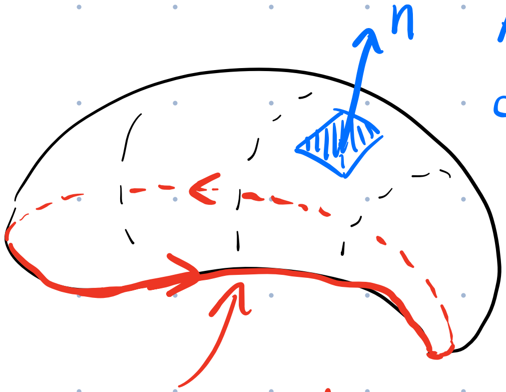
S: integral over an open surface

P: integral over closed boundary of surface

$\hat{n}$

Area element

$d\vec{a} = da\,\hat{n}$



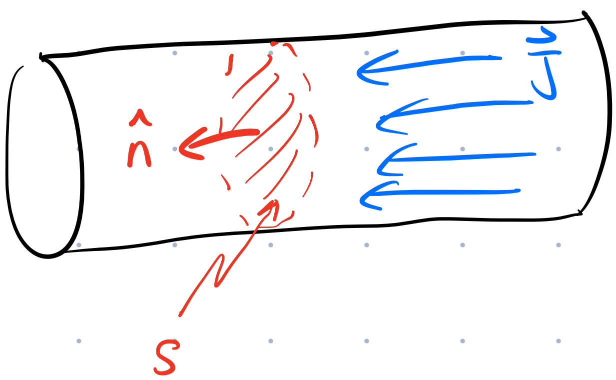
open surface S, in 3-D space

Path P around boundary of surface

Apply Stoke's Th. to Ampère's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

$$\oint_P \vec{B} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 I_{encl}$$



Note that

$$I_{encl} = \int_S \vec{J} \cdot d\vec{a}$$

$$\therefore \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \int_S (\mu_0 \vec{J}) \cdot d\vec{a}$$

$$\therefore \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Last of Maxwell's Eq'ns for static E&M.

Summary

Maxwell's Eq'ns for Electrostatics

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = 0$$

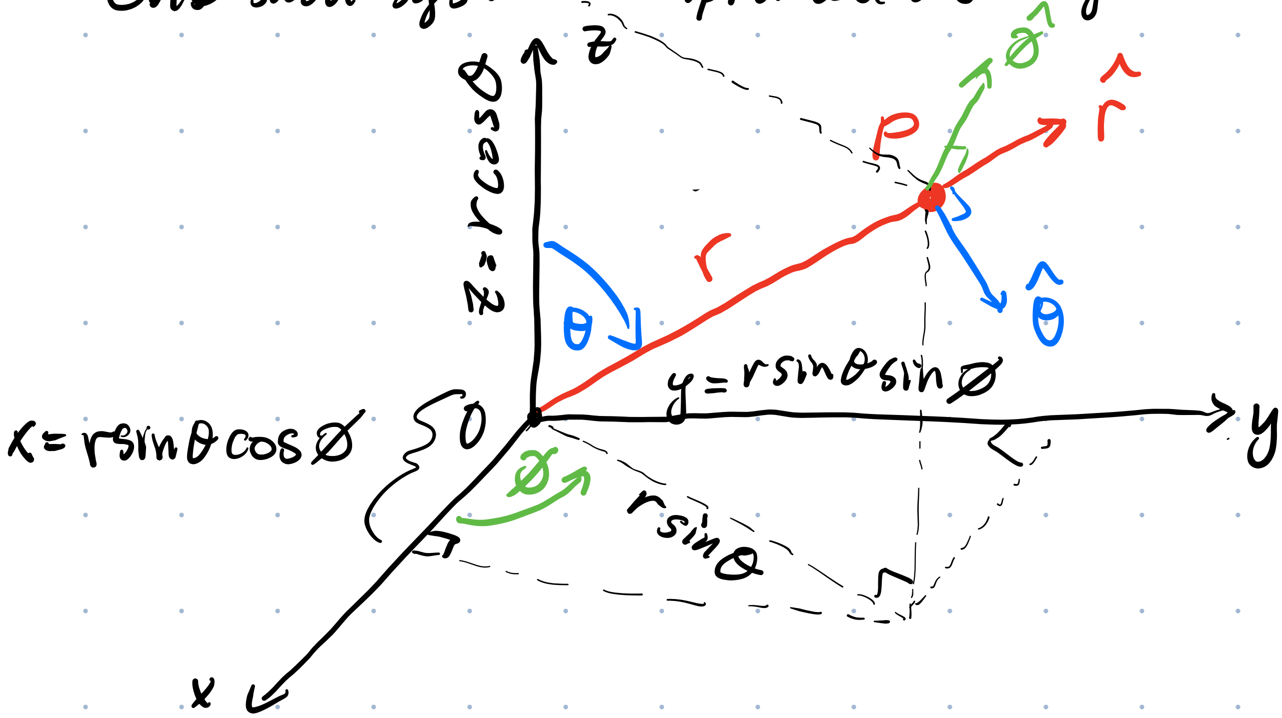
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Griffiths 1.4 Curvilinear Coordinates

Spherical Coordinates.

Rather than specifying the position of a pt. by x, y, z coords., can use a diff. set of coords w/ mutually ⊥ unit vectors.

One such system is spherical coord. sys.



r : dist. from origin O to pt. P

θ : angle between z-axis & radial line to P.

ϕ : angle between x-axis & projection of radial line into xy-plane.

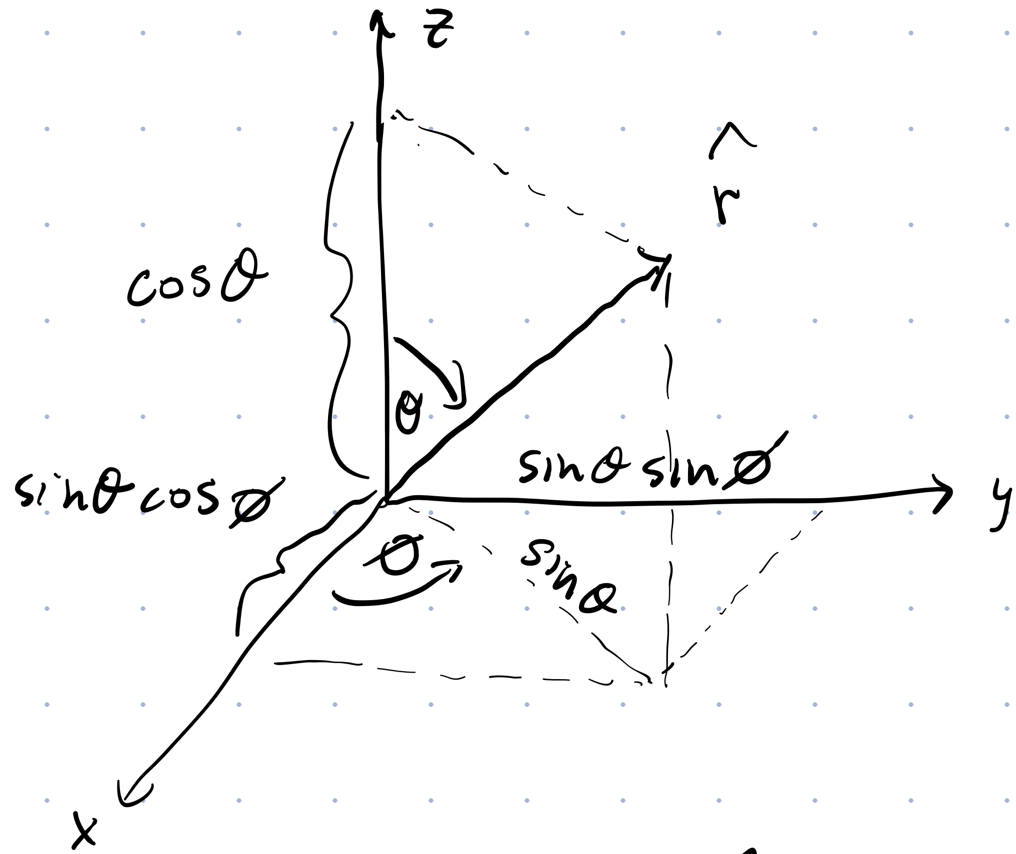
since $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

want to find relationships between x, y, z & r, θ, ϕ.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}.$$

Express $\hat{r}$ in terms of $\hat{x}$, $\hat{y}$, & $\hat{z}$



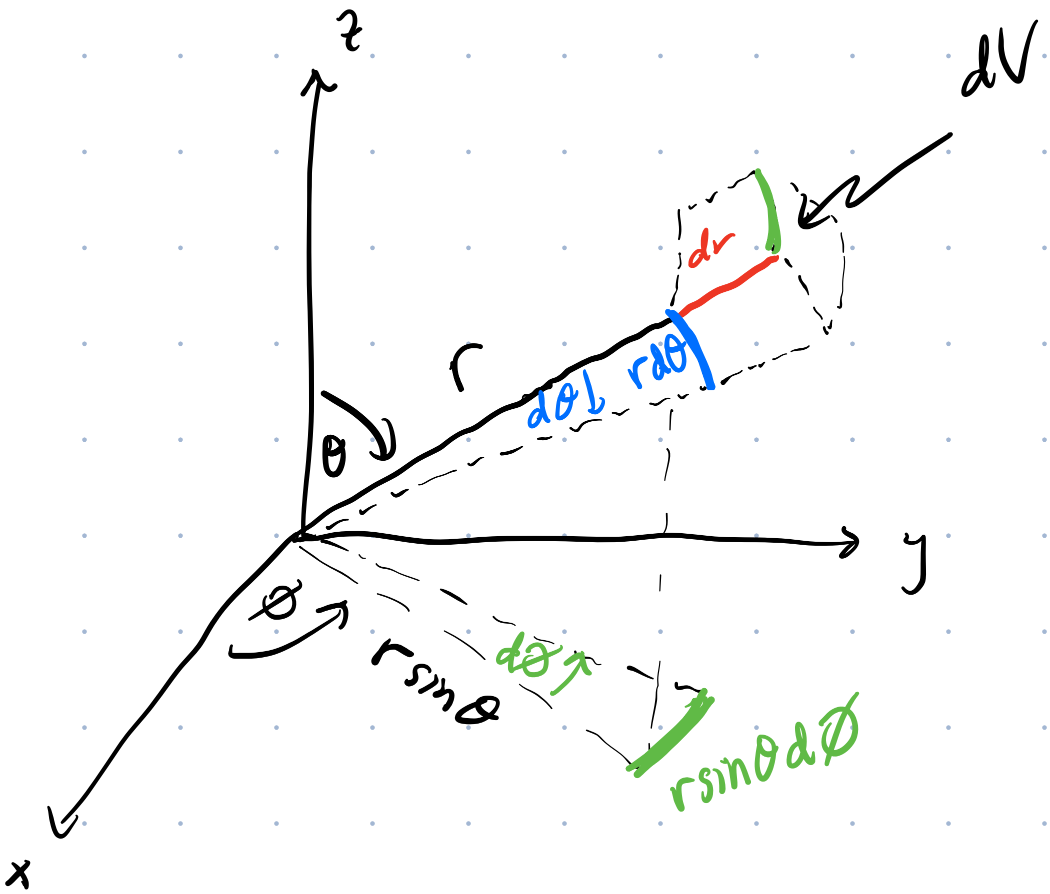
$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

likewise, can show (prob. 1.38) that:

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Consider an infinitesimal volume element in spherical coord.



$$\boxed{\begin{aligned} dV &= (dr)(r\,d\theta)(r\sin\theta\, d\phi) \\ &= r^2 \sin\theta\, dr\, d\theta\, d\phi \end{aligned}}$$

Consider the $\vec{\nabla}$ derivatives in spherical coord.

$$\left(\vec{\nabla}T,\ \vec{\nabla}\cdot\vec{v},\ \vec{\nabla}\times\vec{v},\ \nabla^2 T\right)$$

Start by writting gradient in Cartesian coord.

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

If T is expressed in terms of spherical coord.

s.t. $T = T(r, \theta, \phi)$

just x-deriv.

$$\frac{\partial \underline{T(r,\theta,\phi)}}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial T}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial T}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial T}{\partial \phi}$$

chain rule.

Start w/ $\frac{\partial r}{\partial x}$. know $r = \sqrt{x^2+y^2+z^2}$

$$\therefore \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{r} = \sin\theta\cos\phi$$

$$\boxed{\therefore \frac{\partial r}{\partial x} = \sin\theta\cos\phi}$$ one of the 9 deriv. that we need.

Next, consider $\frac{\partial \theta}{\partial x}$.

First think about $x^2 + y^2$

$$= \underbrace{r^2\sin^2\theta\cos^2\phi}_{x^2} + \underbrace{r^2\sin^2\theta\sin^2\phi}_{y^2}$$

$$= r^2\sin^2\theta\underbrace{(\cos^2\phi + \sin^2\phi)}_{1}.$$

$$\therefore \sin^2\theta = \frac{x^2+y^2}{r^2}$$

$$\therefore \sin\theta = \frac{\sqrt{x^2+y^2}}{r}$$ let's try $\frac{\partial \sin\theta}{\partial x}$

$$\frac{\partial \sin\theta}{\partial x} = \frac{1}{r}\frac{\frac{1}{2}2x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \frac{1}{r}\frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\left(-\frac{1}{r^2}\underbrace{\frac{\partial r}{\partial x}}_{\sin\theta\cos\phi}\right)$$

$$\therefore \cos\theta\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}}{r^2}\sin\theta\cos\phi$$

solve for $\frac{\partial \theta}{\partial x}$

$$x^2+y^2 = r^2\sin^2\theta, \quad x = r\sin\theta\cos\phi$$

$$\therefore \cos\theta\frac{\partial \theta}{\partial x} = \frac{1}{\cancel{r}}\frac{\cancel{r}\cancel{\sin\theta}\cos\phi}{r\cancel{\sin\theta}} - \frac{\cancel{r}\sin\theta}{r^{\cancel{2}}}\sin\theta\cos\phi$$

$$= \frac{\cos\phi}{r} - \frac{\sin^2\theta\cos\phi}{r}$$

$$= \frac{\cos\phi}{r}\left(1-\sin^2\theta\right)$$

$$\therefore \cancel{\cos\theta}\frac{\partial \theta}{\partial x} = \frac{\cos\phi}{r}\cos^{\cancel{2}}\theta$$

$$\boxed{\therefore \frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}}$$

2 of 9 require deriv.

keep going...

If you try $\frac{\partial\phi}{\partial x}$, should find $\boxed{\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}}$

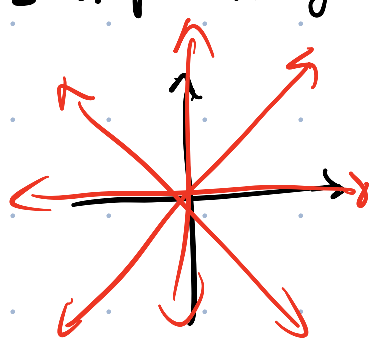
$$\frac{\partial T}{\partial x} = \sin\theta\cos\phi\,\frac{\partial T}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial T}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial T}{\partial\phi}$$

still need $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$ ...

Show $\boxed{\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi}}$

1.5 The Dirac Delta Function

Consider $\vec{v} = \frac{\hat{r}}{r^2}$ ($\propto \vec{E}$ of pt. charge



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In spherical coords. $\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\underbrace{\left(r^2 v_r\right)}_{\text{radial component.}}$

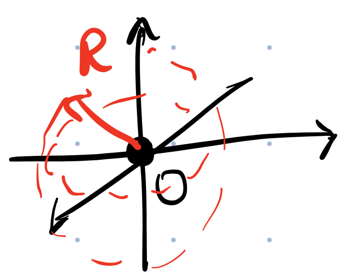
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\underbrace{\left(r^2 \frac{1}{r^2}\right)}_{0.}$$

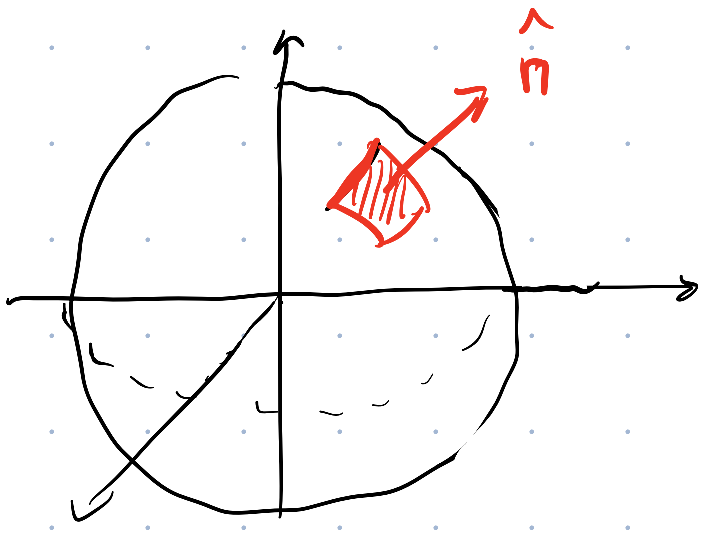
$\therefore \vec{\nabla}\cdot\vec{v} = 0$ everywhere <u>except</u> @ $r=0$ where $\frac{1}{r^2}$ diverges.

Try applying the divergence theorem to $\vec{v}$.

$$\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

Let's do this calc. for a spherical volume of radius R w/ origin at centre.





$d\vec{a} = da\,\hat{n}$

For our spherical volume $\hat{n} = \hat{r}$

$\therefore d\vec{a} = da\,\hat{r}.$

RHS of Stoke's Theorem:

$$\oint_S \vec{v} \cdot d\vec{a} = \oint_S \left(\frac{\hat{r}}{r^2}\right) \cdot \left(da\,\hat{r}\right)$$

$$= \oint_S \frac{da}{r^2}$$

On surface of sphere, $r = R$ is const. &

$da = r^2 \sin\theta\, d\theta\, d\phi$

R

$$= \int_S \frac{\cancel{R^2} \sin\theta\, d\theta\, d\phi}{\cancel{R^2}} = \int_S \sin\theta\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi = \underbrace{\int_{\theta=0}^{\pi} \sin\theta \, d\theta}_{-\cos\theta\Big|_0^{\pi}} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi}$$

$$= 2$$

$$\therefore \oint \vec{v} \cdot d\vec{a} = 4\pi$$

By divergence Theorem:

$$\oint \vec{v} \cdot d\vec{a} = \int_V \underbrace{\vec{\nabla} \cdot \vec{v}}_{0 \text{ every except at origin.}} \, d\tau = 4\pi$$

Define a "generalized" fcn $\delta(\vec{r})$ called the Dirac delta-fcn s.t.

$$\delta(\vec{r}) = \frac{1}{4\pi} \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)$$

or $\quad \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi \, \delta(\vec{r})$

Sub Dirac Delta fcn into Divergence theore above:

$$\int \underbrace{\vec{\nabla}\cdot\vec{v}}_{\vec{\nabla}\cdot\frac{\hat{r}}{r^2}}\, d\tau = \int \cancel{4\pi}\,\delta(\vec{r})\, d\tau = \cancel{4\pi}$$

$$\boxed{\int \delta(\vec{r})\, d\tau = 1}$$