PHYS 301 Sept. 11, 2024

Last Time:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Gradient $\vec{\nabla} T$

Divergence $\vec{\nabla} \cdot \vec{V}$

Curl $\vec{\nabla} \times \vec{V}$

Today: Product Rule for $\vec{\nabla}$ operations (Griffiths 1.2.6)

(i) $\vec{\nabla}(fg)$

(ii) $\vec{\nabla}(\vec{A} \cdot \vec{B})$

(iii) $\vec{\nabla} \cdot (f\vec{A})$

(iv) $\vec{\nabla} \cdot (\vec{A} \times \vec{B})$

(v) $\vec{\nabla} \times (f\vec{A})$

(vi) $\vec{\nabla} \times (\vec{A} \times \vec{B})$

6 possibilities

Griffiths front cover:

$$\nabla(fg) = f(\nabla g) + g(\nabla f)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$ * Griffiths text

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$ # We'll do this one.

$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

Let's try to work on $\vec{\nabla}\cdot(\vec{A}\times\vec{B})$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{\nabla}\cdot\left[\hat{x}(A_yB_z - A_zB_y) - \hat{y}(A_xB_z - A_zB_x) + \hat{z}(A_xB_y - A_yB_x)\right]$$

$$= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) - \frac{\partial}{\partial y}(A_xB_z - A_zB_x) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$$

$$B_z\frac{\partial A_y}{\partial x} + A_y\frac{\partial B_z}{\partial x} - B_y\frac{\partial A_z}{\partial x} - A_z\frac{\partial B_y}{\partial x}$$

$$-A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y} + A_z\frac{\partial B_x}{\partial y} + B_x\frac{\partial A_z}{\partial y}$$

$$+A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - B_x\frac{\partial A_y}{\partial z}$$

$$B_x\underbrace{\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)}_{(\vec{\nabla}\times\vec{A})_x} + B_y\underbrace{\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)}_{(\vec{\nabla}\times\vec{A})_y}$$

x-component of $\vec{\nabla}\times\vec{A}$

$$+ B_z\underbrace{\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)}_{(\vec{\nabla}\times\vec{A})_z} - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$\vec{B}\cdot(\vec{\nabla}\times\vec{A})$

$$\boxed{\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})}$$

1.2.7 Second Derivatives involving $\vec{\nabla}$

Possibilities are:

$\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$

$\vec{\nabla}\cdot(\vec{\nabla}T)$

5 possibilities.

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v})$$

$$\vec{\nabla} \times (\vec{\nabla} T)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v})$$

Below, we consider the possibilities that we'll encounter in PHYS 301.

Start w/ $\vec{\nabla} \cdot (\vec{\nabla} T)$

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

This object is called the Laplacian & it is given the following notation:

$$\nabla^2 T = \vec{\nabla} \cdot (\vec{\nabla} T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Next, consider $\vec{\nabla}\times(\vec{\nabla}T)$

$$\vec{\nabla}\times(\vec{\nabla}T) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix}$$

$$= \hat{x}\underbrace{\left(\frac{\partial}{\partial y}\frac{\partial T}{\partial z} - \frac{\partial}{\partial z}\frac{\partial T}{\partial y}\right)}_{\frac{\partial^2 T}{\partial y\,\partial z} - \frac{\partial^2 T}{\partial z\,\partial y} = 0} - \hat{y}\underbrace{\left(\quad\right)}_{0} + \hat{z}\underbrace{\left(\quad\right)}_{0}$$

$$\boxed{\therefore \vec{\nabla}\times(\vec{\nabla}T) = 0}$$

From electrostatics, we know that $\vec{E} = -\vec{\nabla}V$

$$\therefore \vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}V) = -\vec{\nabla}\times(\vec{\nabla}V) = 0$$

$$\boxed{\therefore \vec{\nabla}\times\vec{E} = 0}$$

one of Maxwell's Eq'ns for <u>Electrostatics</u>

Next, consider

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$$

$$= \vec{\nabla}\cdot\left[\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial}{\partial z}v_y\right)-\hat{y}(\quad)+\hat{z}(\quad)\right]$$

$$= \frac{\partial}{\partial x}\left(\cancel{\frac{\partial v_z}{\partial y}}-\cancel{\frac{\partial v_y}{\partial z}}\right)$$

$$-\frac{\partial}{\partial y}\left(\cancel{\frac{\partial v_z}{\partial x}}-\cancel{\frac{\partial v_x}{\partial z}}\right)$$

$$+\frac{\partial}{\partial z}\left(\cancel{\frac{\partial v_y}{\partial x}}-\cancel{\frac{\partial v_x}{\partial y}}\right)=0$$

Ultimately, we will see that $\vec{B}$ can be written as $\vec{B}=\vec{\nabla}\times\vec{A}$ where $\vec{A}$ is called a vector potential.

$$\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$$

$$\boxed{\vec{\nabla}\cdot\vec{B}=0}$$ Another Maxwell Eq'n.

1.3 Integral Calculus

Common integrals in E&M

Line integrals: $\Delta V = -\int_a^b \vec{E} \cdot d\vec{l}$

closed loop $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$ (Ampere's Law)

Surface integrals: $\Phi = \int_{surface} \vec{B} \cdot d\vec{a}$ (flux)

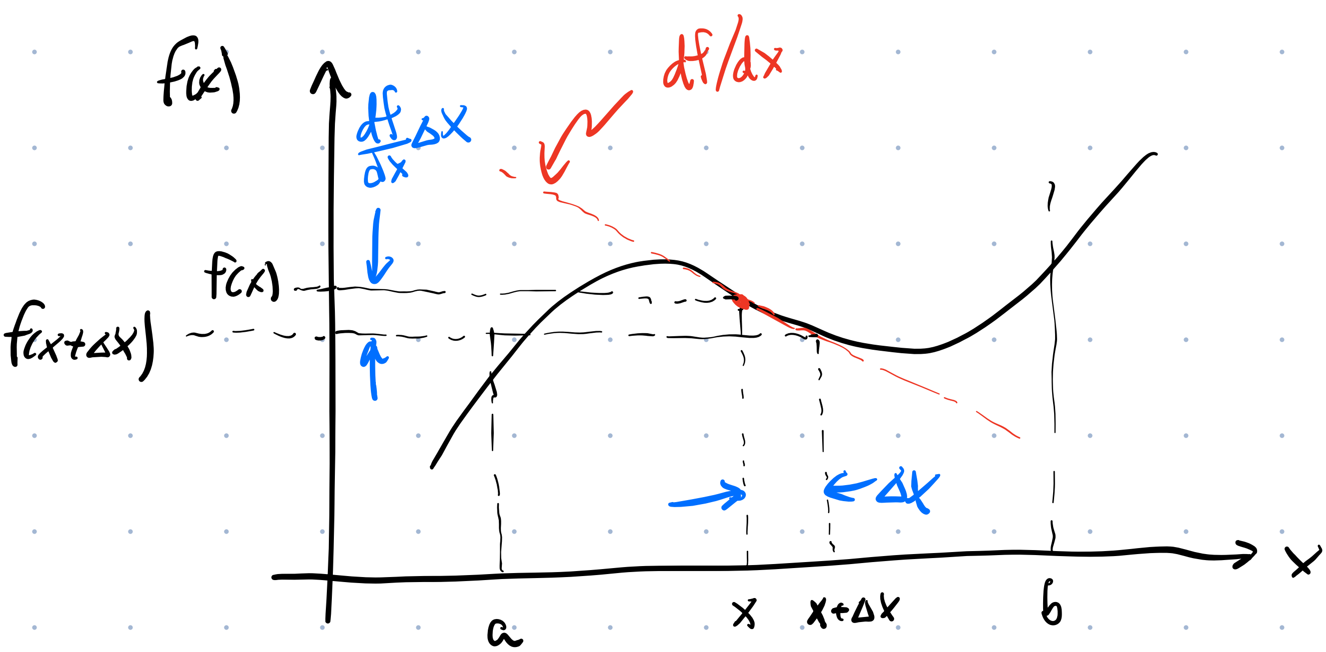
closed surface $\oint \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0}$ (Gauss's Law)

Fundamental Theorem of Calculus

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

Re-express integral as Riemann sum

$$\int_a^b \left(\frac{df}{dx}\right) dx = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \left(\frac{df}{dx}\right) \Delta x$$



$$\therefore \frac{df}{dx}\Delta x \simeq f(x+\Delta x) - f(x)$$

$$\int_a^b \left(\frac{df}{dx}\right) dx = \lim_{\Delta x \to 0} \sum_{a}^{b} \left[f(x+\Delta x) - f(x) \right]$$

$$= \left(\cancel{f(a+\Delta x)} - f(a) \right) + \left(\cancel{f(a+2\Delta x)} - \cancel{f(a+\Delta x)} \right)$$

$$+ \left(\cancel{f(a+3\Delta x)} - \cancel{f(a+2\Delta x)} \right) + \dots$$

$$+ \left(f(b) - \cancel{f(b-\Delta x)} \right)$$

$$\boxed{\therefore \int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)}$$

Fundamental Theorem for Gradients

Similar to above, but applies to scalar fcns of multiple variables $T(x, y, z)$

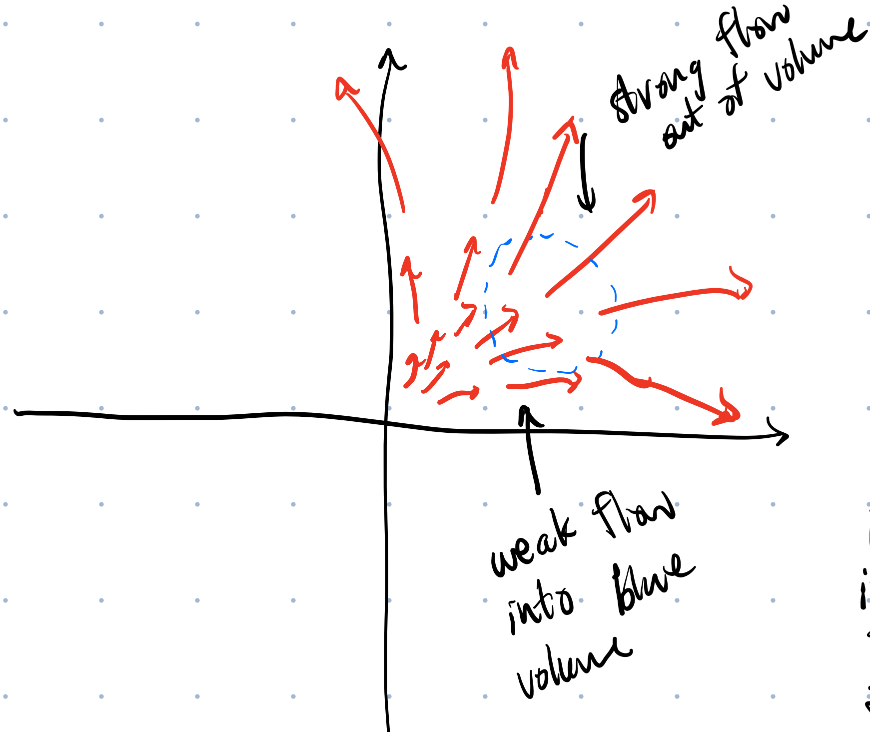
$$dT = \vec{\nabla} T \cdot d\vec{l} = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$\boxed{\int_{\vec{a}}^{\vec{b}} (\vec{\nabla} T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})}$$

pts in 3-D space

Divergence Theorem

One way to determine if a vector field $\vec{v}$ has a non-zero gradient is to examine the "flow" of $\vec{v}$ into/out of an infinitesimal sphere.



if net flow out, then $\vec{\nabla} \cdot \vec{v} > 0$

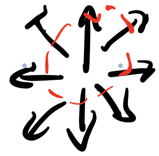
if have net inward flow, $\vec{\nabla} \cdot \vec{v} < 0$

if net flow in/out is zero, then $\vec{\nabla} \cdot \vec{v} = 0$.

$\therefore$ In this example, $\vec{\nabla} \cdot \vec{v} > 0$.

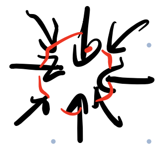
$\vec{\nabla}\cdot\vec{v}$ quantifies the flow of $\vec{v}$ in/out of a region of in space.

source or faucet of $\vec{v}$



$\vec{\nabla}\cdot\vec{v} > 0$

drain or a sink of $\vec{v}$



$\vec{\nabla}\cdot\vec{v} < 0$.

dx dy dz

$$\int_{\text{Volume}} \vec{\nabla}\cdot\vec{v}\, d\tau$$

adds up all sinks & sources of $\vec{v}$ in a volume of space.

Some sinks & sources in the volume will cancell. If there is a net source (sink) then will have field lines exit (enter) the closed surface that surrounds the volume.

∴ We can instead simple track the flux of $\vec{v}$ out (into) this surface.

closed surface surrounding V.

$$\oint_S \vec{v}\cdot d\vec{a}$$

Divergence Theorem

$$\boxed{\int_V \vec{\nabla} \cdot \vec{v} \, d\tau = \oint_S \vec{v} \cdot d\vec{a}}$$

Application to electrostatics . . .

Recall Gauss's Law

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0}$$

Apply divergence Theorem:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \, d\tau = \frac{q_{encl}}{\varepsilon_0}$$

We can express $q_{encl} = \int_V \rho \, d\tau$

charge density

$$\int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

$$= \int_V \left(\frac{\rho}{\varepsilon_0}\right) d\tau$$

$$\boxed{\therefore\ \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$ another Maxwell's eqn.

Gauss's Law :

Integral form $\oint \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0}$

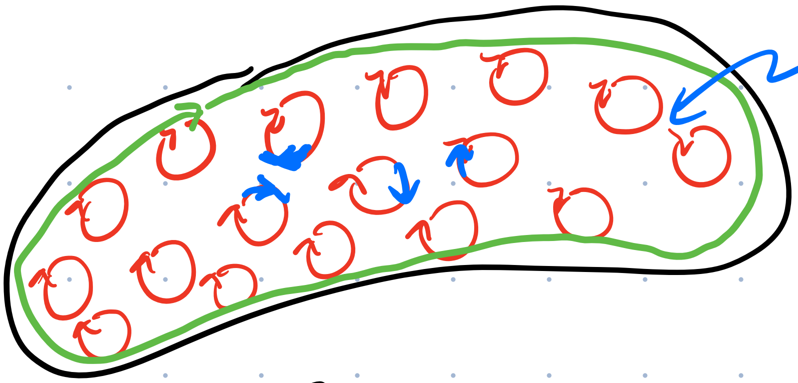
Differential form $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

Stoke's Theorem

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$$

evaluate curl of $\vec{v}$ over surface

finds components of $\vec{v}$ that "circulate" around boundary.



interior contrib. cancel leaving only a net outer "swirl" around boundary.

surface S