PHYS 301 Sept. 9, 2024

Last Time:

Three types of vector multiplication

(i) Scalar mult. $\vec{B} = a\vec{A}$

(ii) Dot Product

$$\vec{A}\cdot\vec{B} = AB\cos\theta = A_{\parallel} B = A B_{\parallel}$$

if $\vec{A} \perp \vec{B}$, $\vec{A}\cdot\vec{B} = 0$

(iii) Cross Product

$$\vec{A}\times\vec{B} = \vec{C}$$

$$|\vec{C}| = |\vec{A}\times\vec{B}| = AB\sin\theta = A_{\perp} B = A B_{\perp}$$

(Area of parallelogram).

if $\vec{A} \parallel \vec{B}$, $\vec{A}\times\vec{B} = 0$

Triple Products:

(i) $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

$= \vec{B} \cdot (\vec{C} \times \vec{A})$

(ii) $\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

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Griffiths 1.2 <u>Differential Calculus.</u>

Gradient: $\vec{\nabla} T$ where $T = T(x, y, z)$

"del"

The change in $T$, which we call $dT$ has 3 possible contributions.

$$dT_x = \frac{\partial T}{\partial x} \Delta x$$

$$dT_y = \frac{\partial T}{\partial y} \Delta y$$

$$dT_z = \frac{\partial T}{\partial z} \Delta z$$

Net change in T for a step in an arbitrary dir'n is given by:

$$\Delta T = \frac{dT}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y + \frac{\partial T}{\partial z} \Delta z$$

Define the "del" operator $\vec{\nabla}$ for notational convenience.

$$\vec{\nabla} \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

If we define an infinitesimal vector displacement

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Consider $\underbrace{\vec{\nabla} T}_{\text{Gradient of } T} \cdot d\vec{l} = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz = \Delta T.$

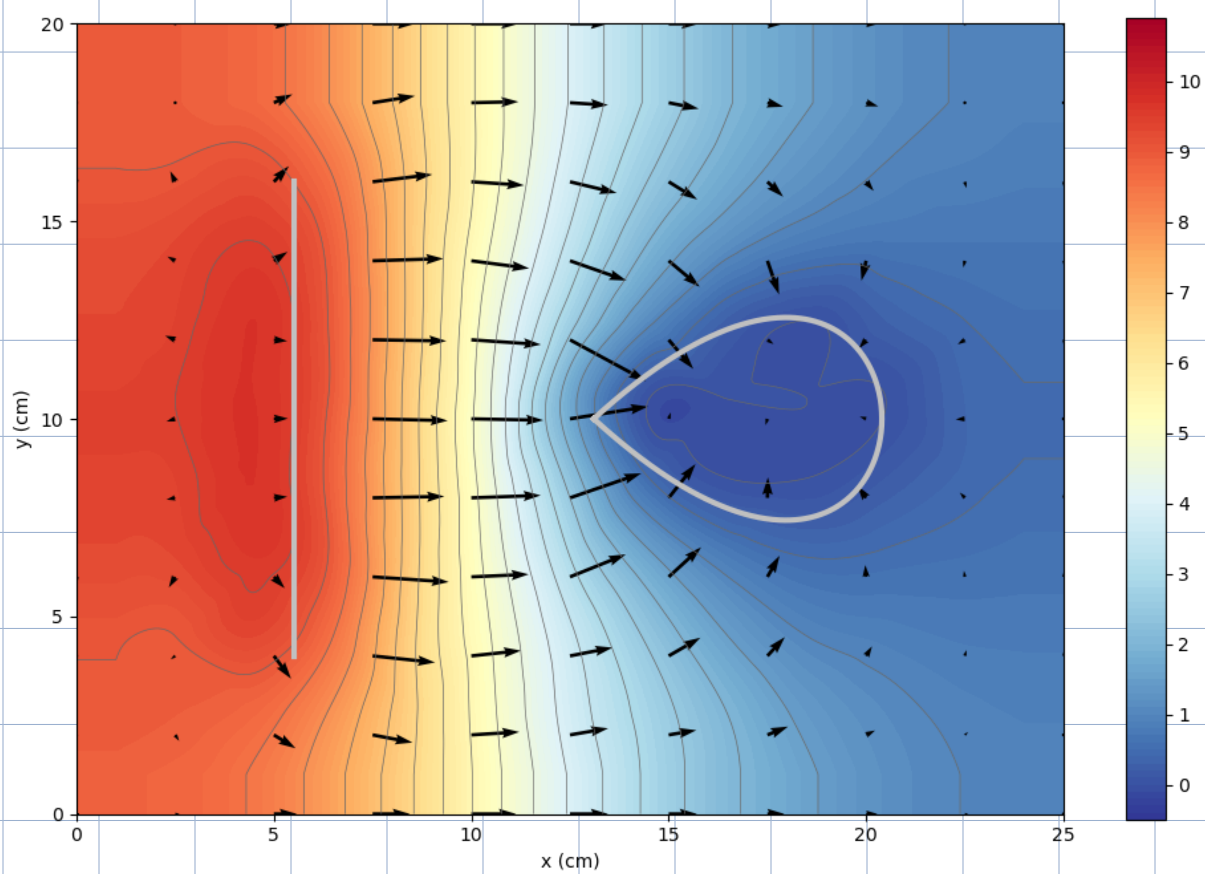
$$\boxed{\therefore dT = \vec{\nabla} T \cdot d\vec{l}}$$

$\therefore dT = |\vec{\nabla} T| \, |d\vec{l}| \cos\theta$

$dT$ is a maximum and positive when $\theta$ is zero.

So if we step in the dir'n of $\vec{\nabla} T$ (i.e. $d\vec{l} \parallel \vec{\nabla} T$), we get the biggest posible change in $T$.

$\boxed{\vec{\nabla} T \text{ points in dirn of greatest increase.}}$



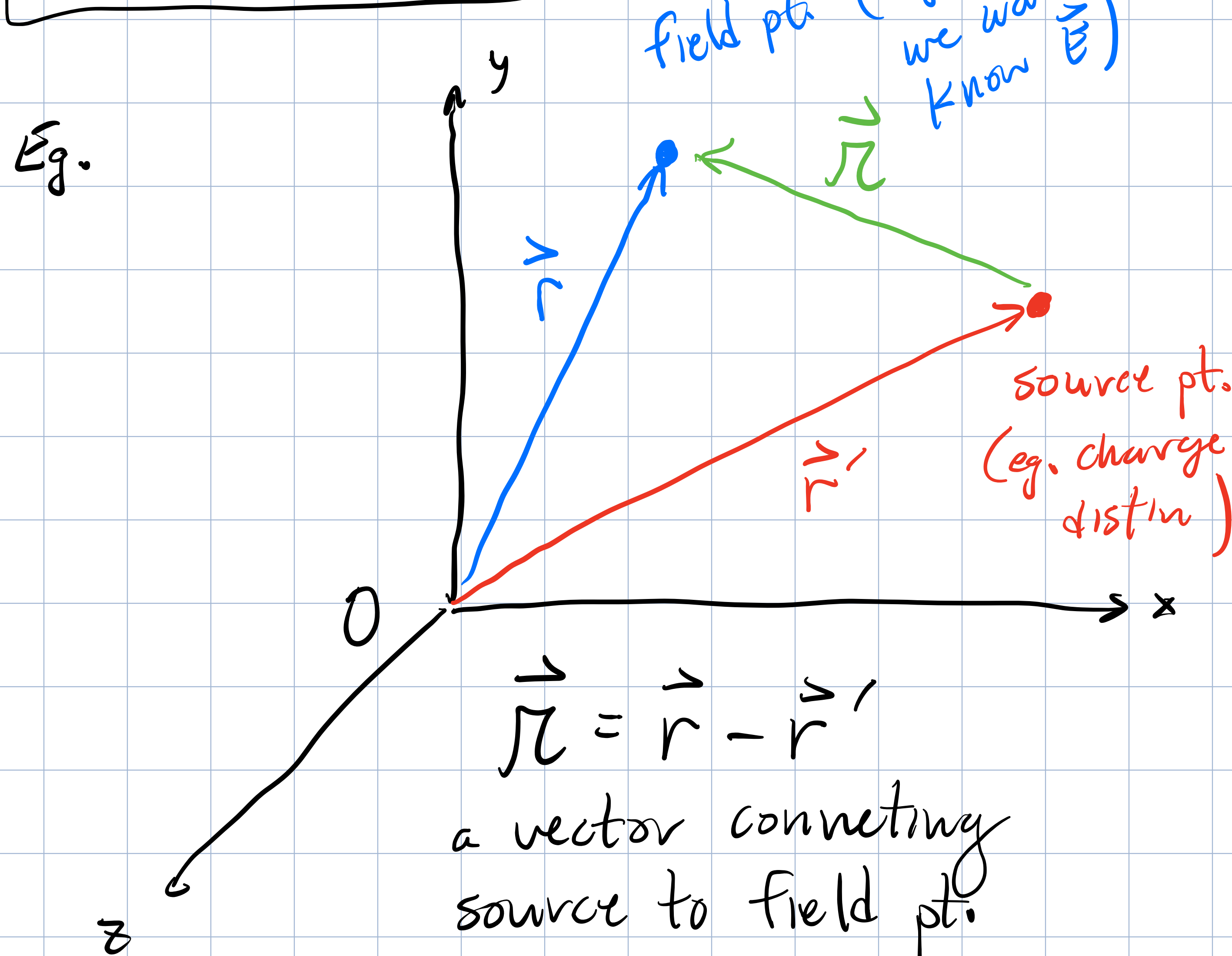

This is an electric potential map (voltage map).
The light lines are equipotentials & the black arrows are $\vec{E}$ field lines.

The dir'n of fastest change in V at any pt. on map is $\perp$ to equipotentials.

Notice that $\vec{E}$ is $\perp$ to equipotentials. But, $\vec{E}$ pts from pos. to neg. V (decreasing V).

$$\boxed{\therefore\ \vec{E} = -\vec{\nabla} V}$$

Eg.




$$\vec{\mathcal{R}} = \vec{r} - \vec{r}\,'$$

a vector connecting source to field pt.

Find $\vec{\nabla}(\mathcal{R}^2)$

Note that $\mathcal{R}^2 = \vec{\mathcal{R}} \cdot \vec{\mathcal{R}}$ is a scalar.

$$\mathcal{R}^2 = (\vec{r} - \vec{r}\,') \cdot (\vec{r} - \vec{r}\,')$$

$$= r^2 + (r')^2 - 2\,\vec{r}\cdot\vec{r}\,'$$

$$= (x^2 + y^2 + z^2) + \left((x')^2 + (y')^2 + (z')^2\right)$$

$$- 2(xx' + yy' + zz')$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}(\mathscr{r}^2) = \hat{x}(2x - 2x')$$

$$+ \hat{y}\,2(y - y') + \hat{z}\,2(z - z')$$

$$\boxed{\therefore \vec{\nabla}(\mathscr{r}^2) = 2(\vec{r} - \vec{r}\,') = 2\vec{\mathscr{r}}}$$

We can also act w/ $\vec{\nabla}$ operator on vectors using dot & cross products.

## Divergence

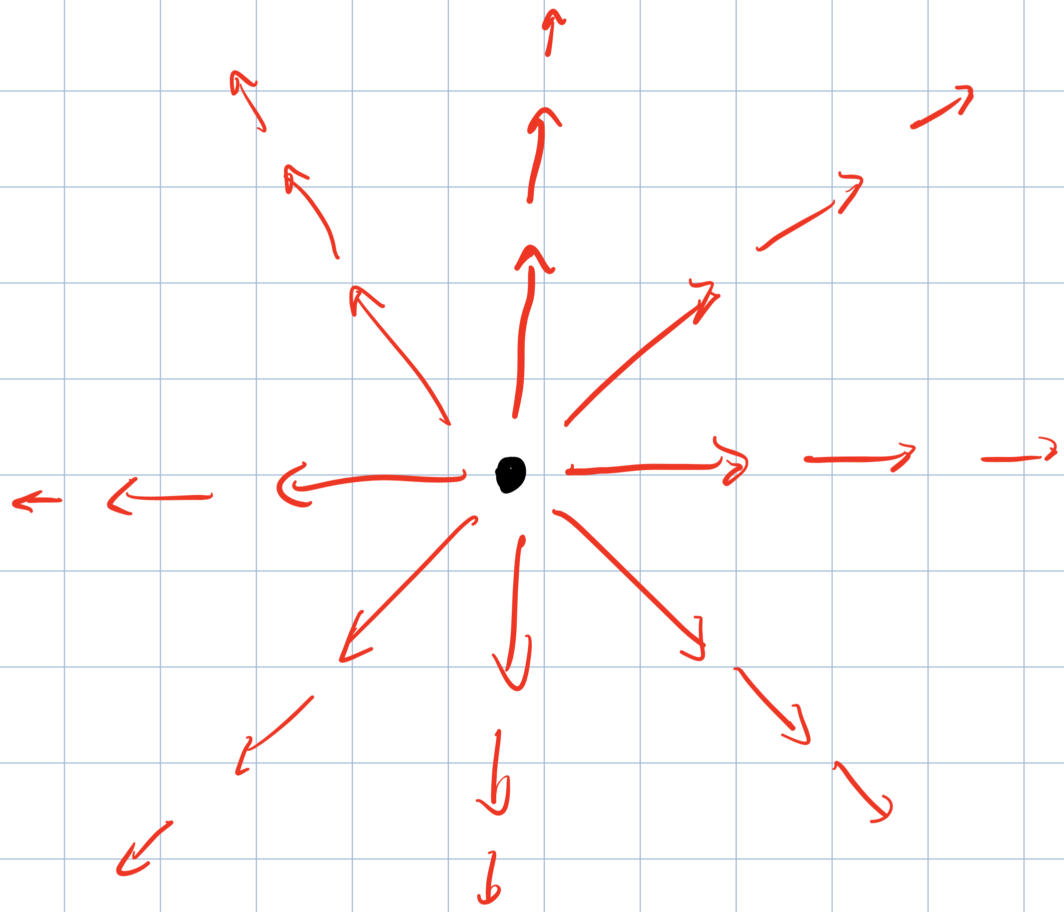
$$\vec{\nabla} \cdot \vec{V}$$

$$\left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \right)$$

$$\therefore \vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Eg. $\vec{E}$ due to a pt. charge:

$$\vec{E} = \frac{k_e q}{r^2} \hat{r}$$



$q > 0$

$$\vec{\nabla}\cdot\vec{E} = \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}\cdot\left(\frac{k_e q}{r^2}\,\hat{r}\right)$$

divergence in spherical coord.
(Front cover of Griffiths)

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

For our pt. charge $E_\theta = E_\phi = 0$

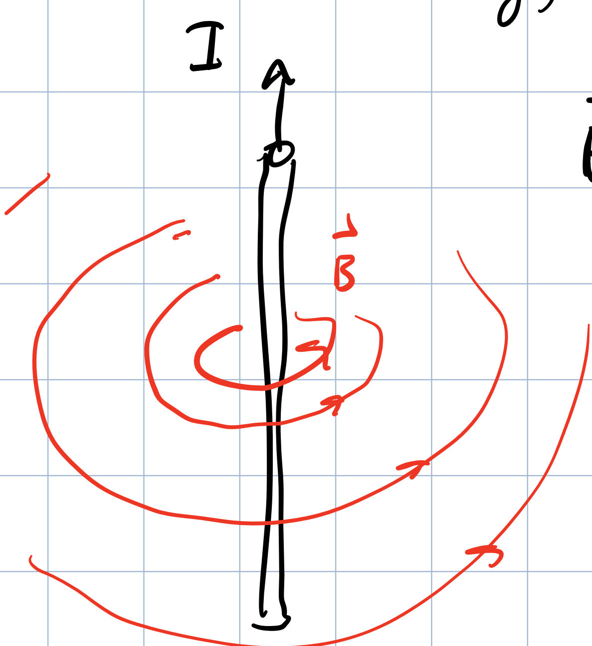
$$E_r = \frac{k_e q}{r^2}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left( \cancel{r^2} \frac{k_e q}{\cancel{r^2}} \right) = 0 \,!$$

2. Curl of a vector.

$$\vec{\nabla} \times \vec{v}$$

Eg. Consider the magnetic field $\vec{B}$ due to a long, straight current.




$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

In cylindrical coord.

$$\vec{\nabla}\times\vec{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r}$$

$$+ \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

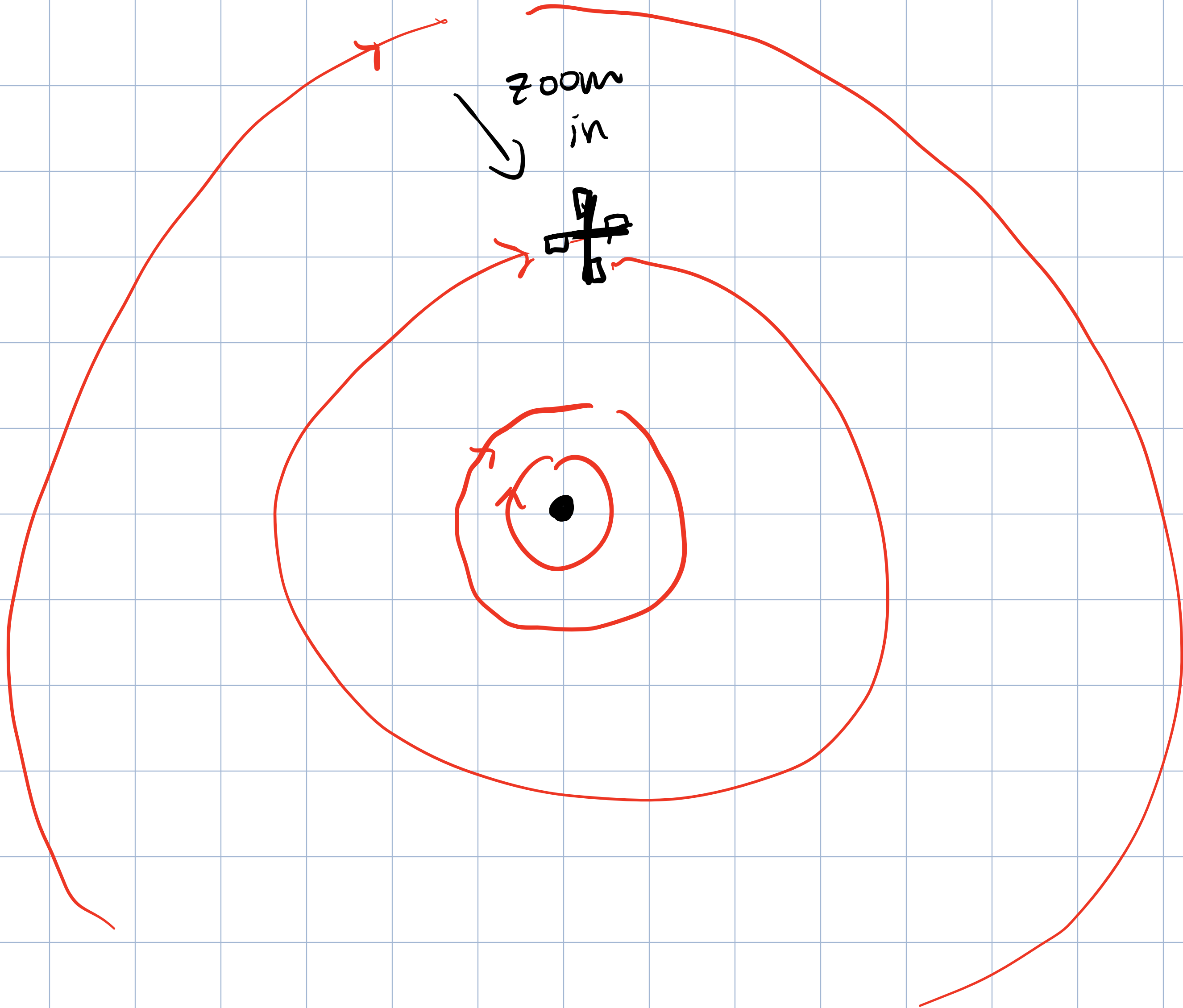
$$\vec{B} = B_\phi\,\hat{\phi} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

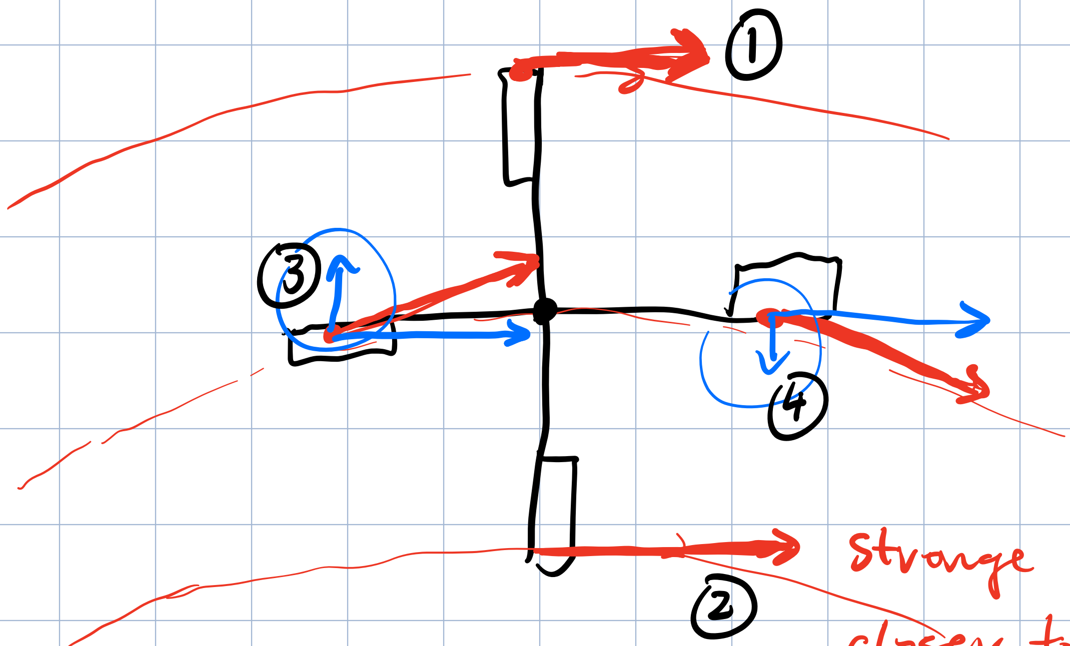
$$\vec{\nabla}\times\vec{B} = -\cancelto{0}{\frac{\partial B_\phi}{\partial z}}\hat{r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r B_\phi\right)\hat{z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\,\frac{\mu_0 I}{2\pi \cancel{r}}\right)\hat{z} = 0!$$

zero curl everywhere except

@ $r=0$ where $\vec{B}$ diverges.




strong b/c closer to current

Here, paddle doesn't spin b/c ③ & ④ have rotations that extactly cancel the rotations due to ① & ②.