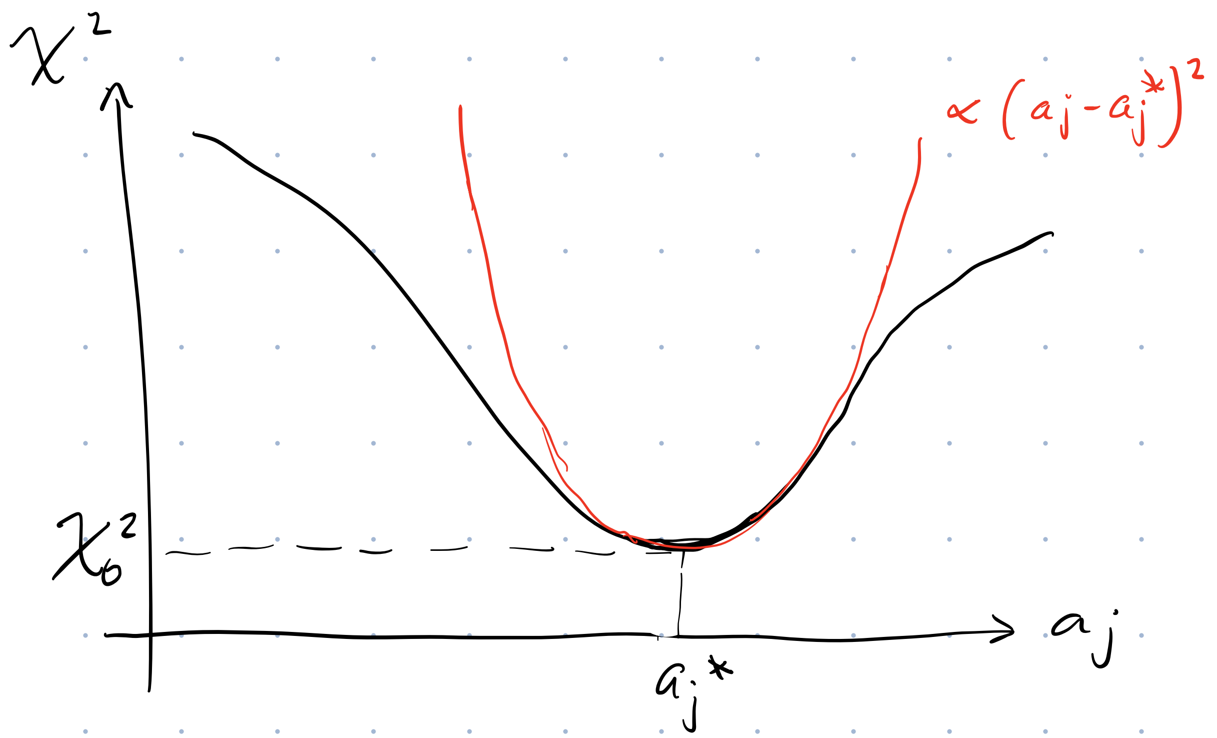
PHYS 232 April 2, 2025

Due dates:

- Assignment #6 : Today
- Experiment #5 lab notebooks : Fri. Apr. 4 @ 09:00
  - Bring lab notebooks to lecture
- Formal Report : Mon. Apr. 7 @ 14:00
  - Bring printed and stapled report to SCI 241.

Last Time:

Near its minimum, χ^2 varies quadratically w.r.t. changes in the a_j parameter values



$$\chi^2 \approx \chi_0^2 + B_j(a_j - a_j^*)^2$$

By finding χ^2 at 3 pts. near the minimum, can find:

$$\boxed{B_j = \frac{\chi_1^2 - 2\chi_2^2 + \chi_3^2}{2(\Delta a_j)^2}}$$

The uncertainty in parameter a_j is given by

$$\sigma_j = \frac{1}{\sqrt{B_j}}$$

$\therefore$ best-fit value for a_j is:

$$\boxed{a_j = a_j^* \pm \frac{1}{\sqrt{B_j}}}$$

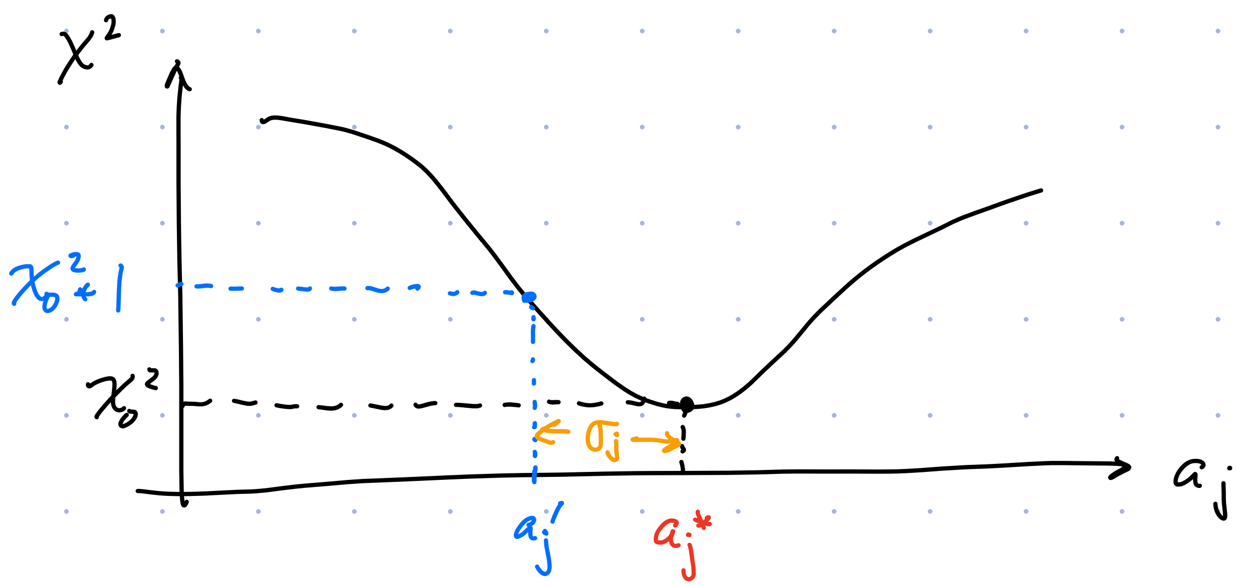
<u>Today</u> Develop some intuition about meaning of χ^2.

Since near its minimum $\chi^2 = B_j (a_j - a_j^*)^2 + \chi_0^2$

& $\sigma_j^2 = \frac{1}{B_j}$

$$\therefore \chi^2 = \frac{(a_j - a_j^*)^2}{\sigma_j^2} + \chi_0^2 \quad (\text{near min.})$$

Consider sitting at $a_j = a_j^*$ & then moving away from this minimum position where $\chi^2 = \chi_0^2$:



As we move away from a_j^* (in either dir'n) χ^2 increases.
Imagine changing a_j until χ^2 goes from

χ_0^2 to χ_0^2+1. i.e. increase χ^2 above its minimum value by 1. The pt. at which χ^2 goes to χ_0^2+1 we will denote as a_j'.

Know $\chi^2 = \dfrac{(a_j - a_j^*)^2}{\sigma_j^2} + \chi_0^2$

now, take $a_j = a_j'$ s.t. $\chi^2 = \chi_0^2 + 1$

$$\therefore \cancel{\chi_0^2} + 1 = \frac{(a_j' - a_j^*)^2}{\sigma_j^2} + \cancel{\chi_0^2}$$

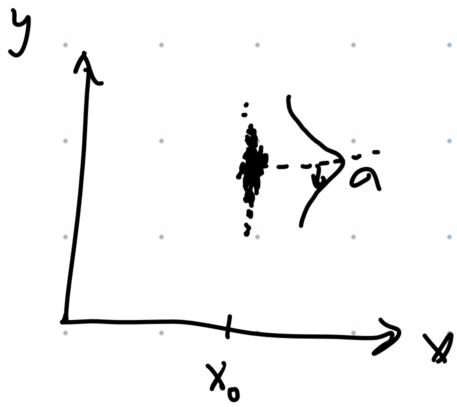
$$\therefore \sigma_j^2 = (a_j' - a_j^*)^2$$

$$\boxed{\sigma_j = |a_j' - a_j^*|}$$

If we move away from the minimum of χ^2 along the a_j axis s.t. $\chi^2 \to \chi_0^2+1$, then the distance moved $|a_j' - a_j^*|$ is an estimate of the uncertainty in a_j^*

What should we expect for a value of χ^2 after a fit to experimental data?

Definition of χ^2 is: $\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$



On average, we expect the deviation $y_i - y(x_i)$ to be approximately equal to $\pm \sigma_j$.

$$\left\{ \sigma_j = \sqrt{\frac{1}{N-1} \sum \underbrace{(x_i - \mu)^2}_{y_i - y(x_i)}} \right\}$$

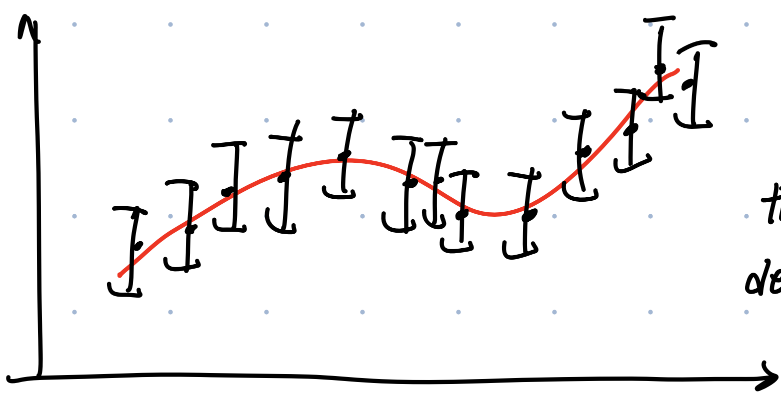
On average, expect:

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2 \approx \sum_{i=1}^{N} \left(\frac{\pm \sigma_i}{\sigma_i} \right)^2$$

$$= \sum_{i=1}^{N} 1 = N$$

- For N large, a good fit to a data set should yield $\chi^2 \approx N$.

- If you find that χ^2 is significantly less than N, it prob. means that σ_i has been over estimated.

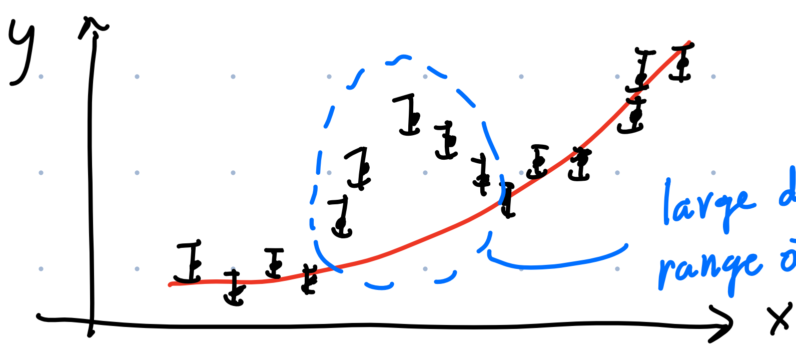


Here, error bars (σ_i) are much bigger than the typical deviation $y_i - y(x_i)$

$\Downarrow$

$\chi^2 \ll N$

- If χ^2 is larger than N, it could be an indication that your model does not capture all of the features present in the data.



large deviations in this range of x-values. → leads to $\chi^2 > N$.

Overall $\chi^2 > N$ → indicates a poor fit to data assuming reasonable estimates of σ_i have been made.

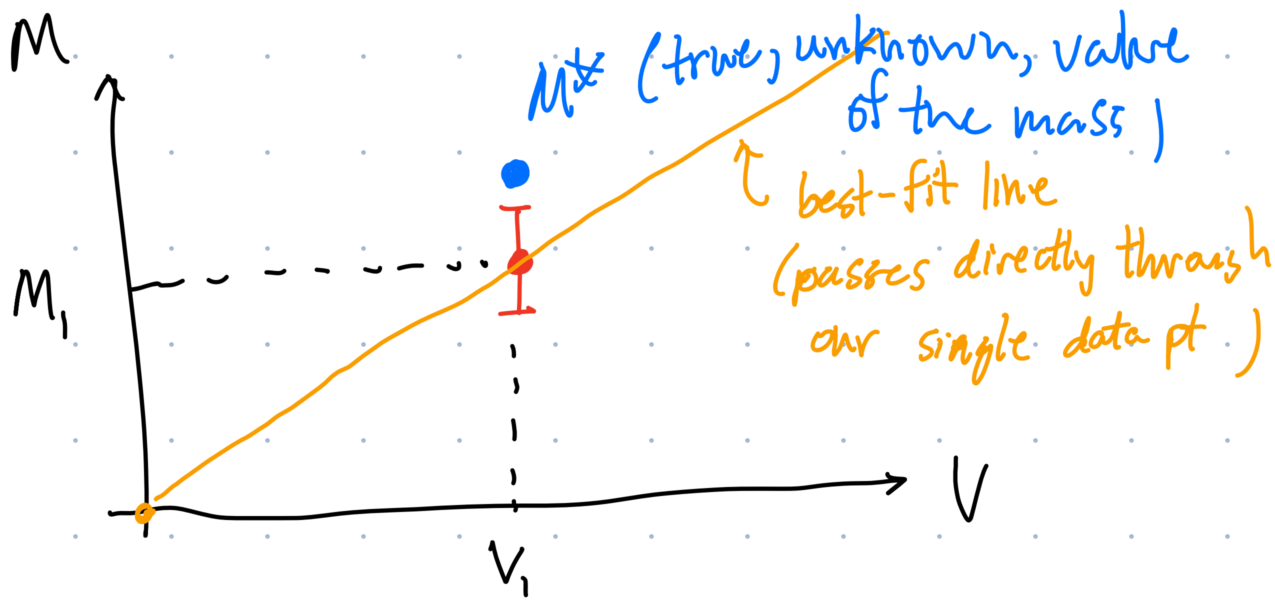
→ χ^2 is a "goodness of fit" parameter when σ_i estimated appropriately.

Expecting $\chi^2 \approx N$ is correct when N is large. However, consider the following extreme case:

Meas. the volume of an object & meas. its mass. Then, determine the density ρ from a "fit" to the data.

$$\underbrace{M}_{y} = \underbrace{\rho}_{\text{slope}} \underbrace{V}_{x}$$

1-parameter fit (ρ) w/ $N=1$ data pts.



If we have only a single pt to fit to, the deviation $y_i - y(x_i) = 0$ & $\chi^2 = 0$.

Since our best-fit line is biased to pass exactly through our single measurement, the calculated χ^2 value is artificially low ($\chi^2 = 0$).

If we could calculate χ^2 using the <u>true</u> value of M (called M* in figure) then we would get $\chi^2 \approx 1$ ($= N$).

Resolution is that expected value of χ^2 is not N, but is $N-m$ where m is the number of parameters in your fit.

For $N=1$ & a one-parameter fit (slope or density ρ) we expect

$$\chi^2 = \underset{\text{no. of data pts}}{N} - \underset{\text{no. of fit parameters.}}{m} = 1-1=0 \checkmark$$

χ^2 summary:

- Expect $\chi^2 = N-m$ for a good fit w/ reasonable estimates of σ_i.

Usually call $N-m=\nu$ "no. of degrees of freedom".

- $\chi^2 > N-m=\nu$ indicates a poor fit if σ_i are reasonable.

If $\chi^2 < N-m = \nu$, then prob. over-estimate of the σ_i values.

Reduced χ^2 is denoted

$$\chi_\nu^2 \equiv \frac{\chi^2}{\nu}$$

For a good fit w/ reasonable σ_i, expect

$$\boxed{\chi_\nu^2 \simeq 1}$$