PHYS 232 March 26, 2025

## Due dates:

- Assignment #6 : Wed. Apr. 2 @ 09:00
- Experiment #5 lab notebooks : Fri. Apr. 4 @ 09:00
  - Bring lab notebooks to lecture
- Formal Report : Mon. Apr. 7 @ 14:00
  - Bring printed and stapled report to SCI 241.

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## Last Time:

Fit data to :

$$y(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots + a_m f_m(x)$$

Meas. $(x_i, y_i \pm \sigma_i)$ for $i = 1 .. N$

Want to determine :

$$a_1 \pm \sigma_{a_1}, \; a_2 \pm \sigma_{a_2}, \; \ldots, \; a_m \pm \sigma_{a_m}$$

We found the $a_k$ values from

$$\boxed{\underline{a} = \underline{\underline{\varepsilon}}\,\underline{\beta}}$$

Diagonal elements of $\underline{\underline{\varepsilon}}$ give the square of the uncertainty in the best-fit parameters

$$a_1 \pm \sqrt{\varepsilon_{11}}$$
$$a_2 \pm \sqrt{\varepsilon_{22}}$$
$$\vdots$$
$$a_m \pm \sqrt{\varepsilon_{mm}}$$

Today: Nonlinear Fits

Examine the case of a non-linear fit,

i.e. $y(x)$ is not of the form $\sum_{k=1}^{m} a_k f_k(x)$

Eg. $y(x) = a_1 \sin(a_2 x)$

Compare & contrast linear & non-linear fits.

linear case

$$y = a_1 + a_2 x$$

of the form $\sum_k a_k f_k(x)$

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - a_1 - a_2 x_i}{\sigma_i} \right)^2$$

minimize w.r.t. $a_1$ & $a_2$

$$\frac{\partial \chi^2}{\partial a_1} = 2 \sum \left( \frac{y_i - a_1 - a_2 x_i}{\sigma_i} \right) \left( -\frac{1}{\sigma_i} \right) = 0$$

$$\therefore \sum \frac{y_i}{\sigma_i^2} = \sum \frac{a_1 + a_2 x_i}{\sigma_i^2} \quad ①$$

no $a_1$ or $a_2$

$$\frac{\partial \chi^2}{\partial a_2} = 2 \sum \left( \frac{y_i - a_1 - a_2 x_i}{\sigma_i} \right) \left( \frac{-x_i}{\sigma_i} \right) = 0$$

$$\sum \frac{x_i y_i}{\sigma_i^2} = \sum \frac{x_i}{\sigma_i^2} (a_1 + a_2 x_i) \quad ②$$

no $a_1$ or $a_2$

non-linear case

$$y = a_1 \sin(a_2 x)$$

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - a_1 \sin(a_2 x_i)}{\sigma_i} \right)^2$$

minimize w.r.t. $a_1$ & $a_2$

$$\frac{\partial \chi^2}{\partial a_1} = 2 \sum \frac{y_i - a_1 \sin(a_2 x_i)}{\sigma_i} \cdot \left( -\frac{\sin(a_2 x_i)}{\sigma_i} \right) = 0$$

$$\therefore \sum \underbrace{\frac{y_i \sin(a_2 x_i)}{\sigma_i^2}}_{\text{involves } a_2} = \sum \frac{a_1 \sin^2(a_2 x_i)}{\sigma_i^2} \quad ①'$$

$$\frac{\partial \chi^2}{\partial a_2} = 2 \sum \frac{y_i - a_1 \sin(a_2 x_i)}{\sigma_i} \cdot \left( -\frac{x_i a_1}{\sigma_i} \cos(a_2 x_i) \right) = 0$$

involves $a_1$ & $a_2$

$$\sum \frac{x_i y_i}{\sigma_i^2} a_1 \cos(a_2 x_i) \qquad (2')$$

$$= \sum \frac{x_i a_1^2}{\sigma_i^2} \cos(a_2 x_i) \sin(a_2 x_i)$$

For the linear fit, can express (1) & (2) as a matrix eq'n w/ $a_k$ parameters in a single column matrix

$$\underbrace{\begin{pmatrix} \sum \frac{y_i}{\sigma_i^2} \\ \sum \frac{x_i y_i}{\sigma_i^2} \end{pmatrix}}_{\underline{\beta}} = \underbrace{\begin{pmatrix} \sum \frac{1}{\sigma_i^2} & \sum \frac{x_i}{\sigma_i^2} \\ \sum \frac{x_i}{\sigma_i^2} & \sum \frac{x_i^2}{\sigma_i^2} \end{pmatrix}}_{\underline{\underline{\alpha}}} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\underline{a}}$$

$$\therefore \underline{a} = \underline{\underline{\alpha}}^{-1} \underline{\beta}$$

For non-linear fit, the $a_k$'s in (1') & (2') are on both sides of the eq'ns & $a_2$ is inside a trig. fcn. Cannot write down an equiv. matrix eq'n for this case.

$\Rightarrow$ need a new fitting method.

## One method for non-linear fits:

Objective is still to minimize

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

For example, want to minimize:

$$\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - a_1 \sin(a_2 x_i)}{\sigma_i} \right)^2$$

Strategy:

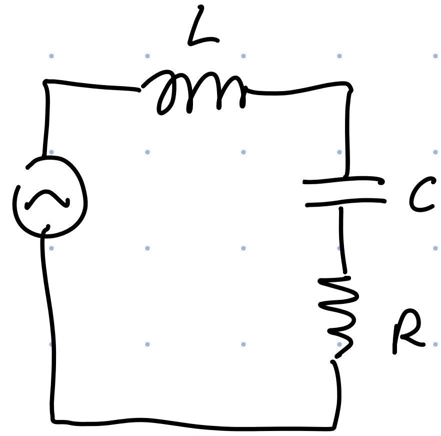
1. Pick a rangle of possible values for $a_1$ & $a_2$.

$$a_{1,min} \leq a_1 \leq a_{1,max}$$
$$a_{2,min} \leq a_2 \leq a_{2,max}$$

have to know something about physical system to select reasonable ranges.

Aside

Eg: LRC Circuit.

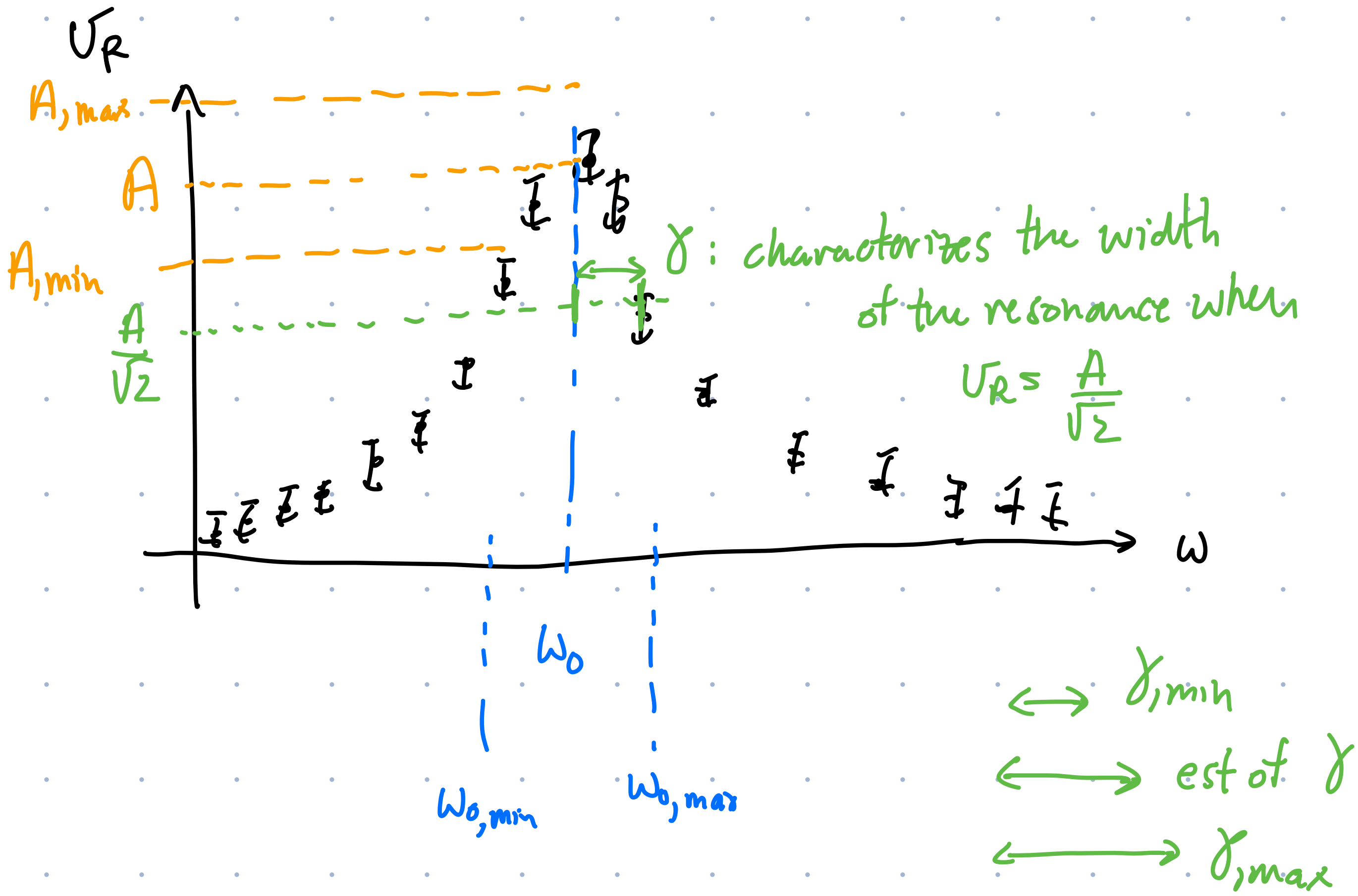



Voltage across R in

$$V_R = \frac{A}{\sqrt{1 + \frac{\omega^2}{\gamma^2}\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}}$$

meas. $V_R$ vs $\omega$ → fit parameters are: $A, \gamma, \omega_0$

Need to use the meas. data to est. reasonable ranges for the fit parameters.




2. Pick a step size $\Delta a_1$ & $\Delta a_2$ for each parameter.

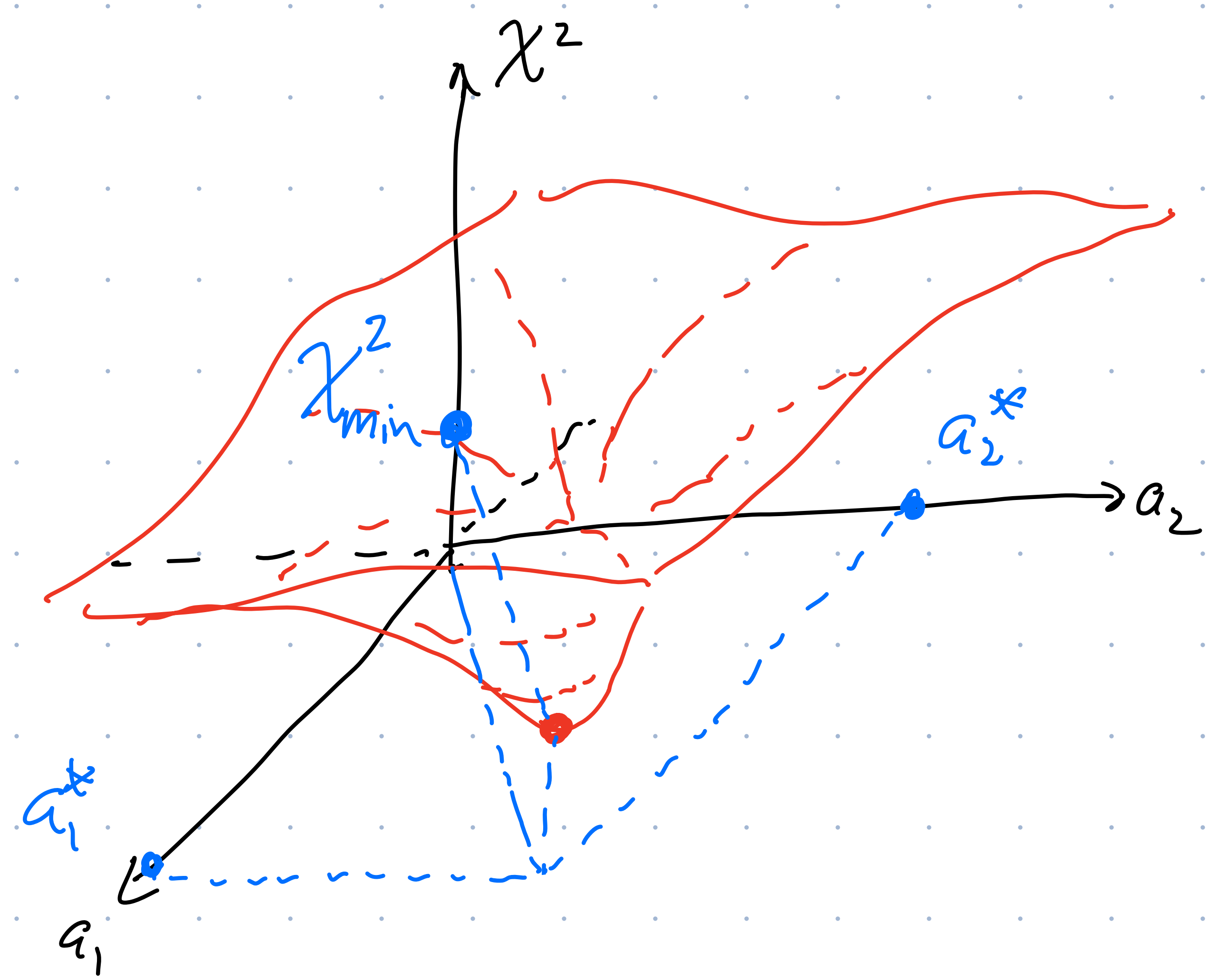
- small $\Delta a_1$, $\Delta a_2$ good for high resolution, but takes longer to calc. all $\chi^2$ values.

3. Evaluate $\chi^2$ at all possible combos of $(a_1, a_2)$.

$a_{1,min}$, $a_{1,min} + \Delta a_1$, $a_{1,min} + 2\Delta a_1$, ...

$a_{2,min}$, $a_{2,min} + \Delta a_2$, $a_{2,min} + 2\Delta a_2$, ...

Best-fit parameters are the values of $a_1$ & $a_2$ that give the minimum for $\chi^2$.




Plot the $\chi^2$ surface vs $a_1$ & $a_2$

If $(a_1^*, a_2^*)$ give the minimum value of $\chi^2$, then they are our estimates of the best-fit parameters.

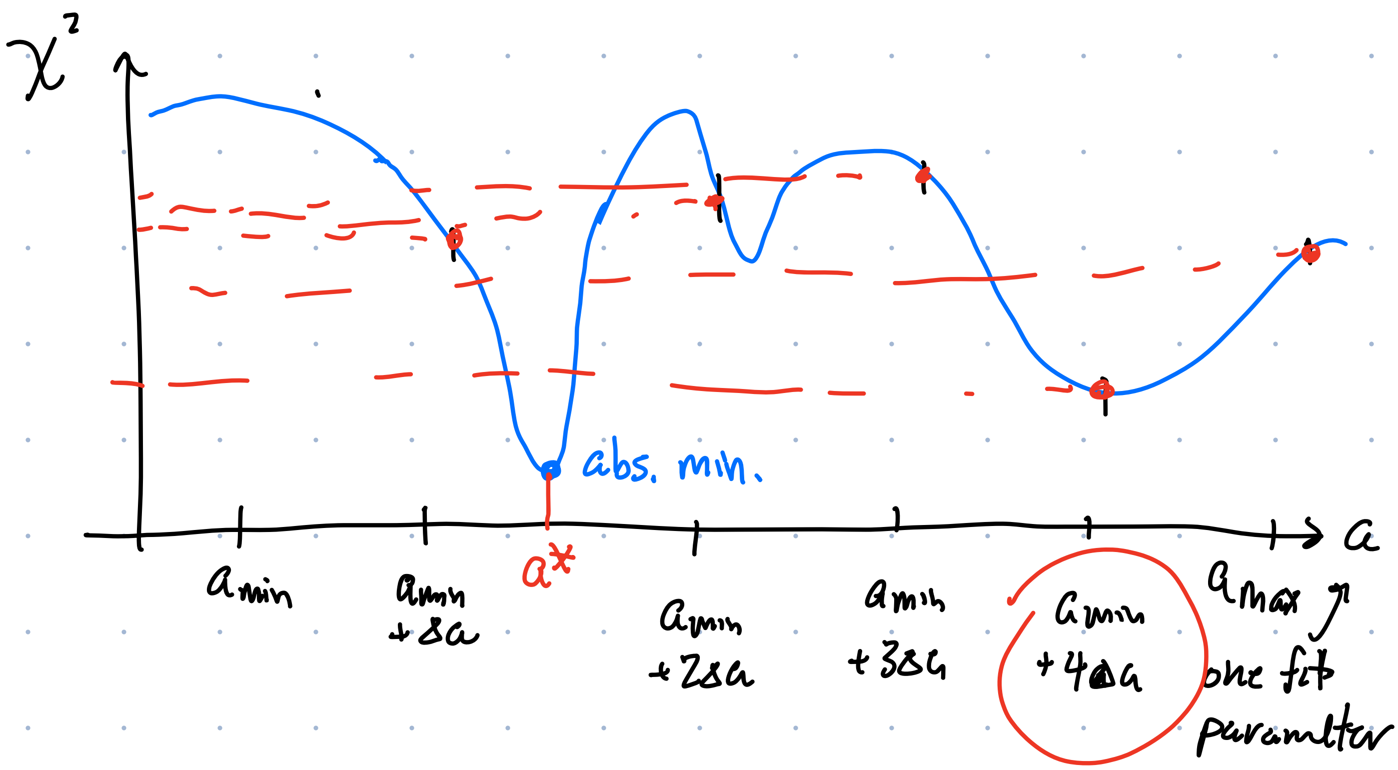
Another way to visualize the same thing:

| $a_2$ \ $a_1$ | $a_{1,min}$ | $a_{1,min} + \Delta a_1$ ($a_1^*$) | $a_{1,min} + 2\Delta a_1$ | ... | $a_{1,max}$ |
|---|---|---|---|---|---|
| $a_{2,min}$ | 36 | 42 | 41 | | 68 |
| $a_{2,min} + \Delta a_2$ | 28 | 36 | 34 | | 90 |
| $a_2^*$ $a_{2,min} + 2\Delta a_2$ | 22 | 18 | 24 | | 36 |
| ⋮ | | | | | |
| $a_{2,max}$ | 50 | 49 | 62 | | 112 |

$\chi^2_{min}$

populate table w/ $\chi^2$ values.

If the non-linear fcn is complicated & changing rapidly, it is possible to miss the absolute global minimum.




In This case, would identify $a_{min} + 4\Delta a$ as our best-fit parameter, missing $a^*$ b/c of our coarse search.