PHYS 232 March 14, 2025

- Assignment #5 is on course website this weekend

- Start working on your formal report
  - due April 7 @ 14:00 in SCI 241

<u>Last Time:</u> given a linear set of data $(x_i, y_i \pm \sigma_i)$, the best intercept & slope for the line passing through the data are given by:

$$\boxed{a = \frac{1}{\Delta}\left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2}\right) \qquad \Delta = \sum \frac{1}{\sigma_i^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2}$$

intercept

$$\boxed{\sigma_a^2 = \frac{1}{\Delta} \sum \frac{x_i^2}{\sigma_i^2}}$$

$$\boxed{b = \frac{1}{\Delta}\left(\sum \frac{1}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2}\right)}$$

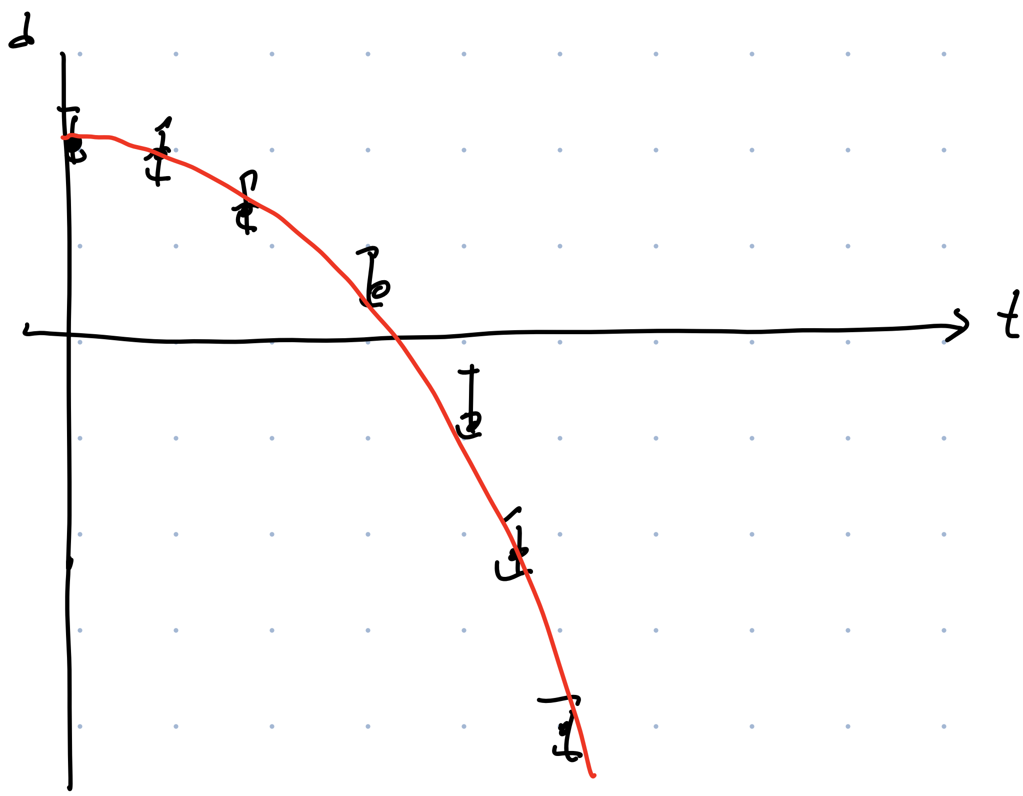
slope.

$$\boxed{\sigma_b^2 = \frac{1}{\Delta} \sum \frac{1}{\sigma_i^2}}$$

Today: Linearizing data

Eg. Object in free starting rest

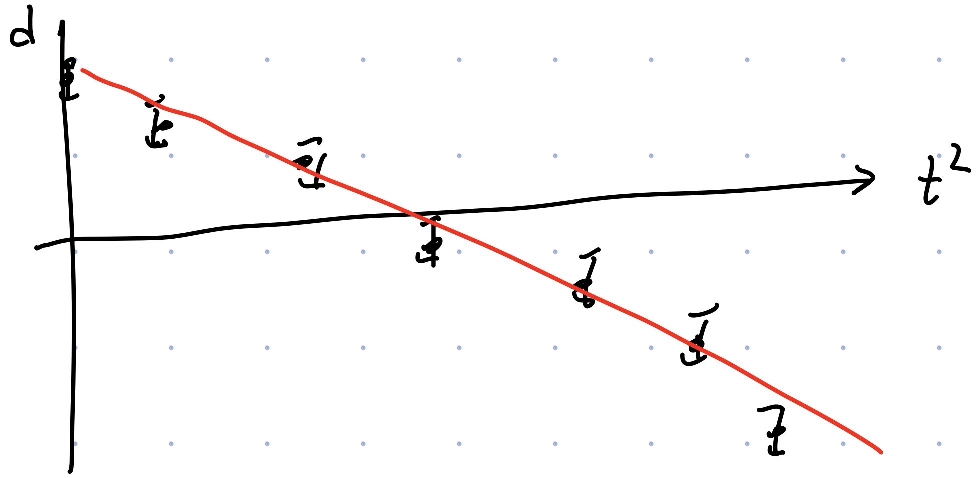
$$d = d_0 - \frac{1}{2} g t^2$$



Instead of plotting d vs t consider

d vs t^2

$$\underset{y}{d} = \underset{a}{d_0} - \underbrace{\tfrac{1}{2} g}_{b} \underbrace{(t^2)}_{x}$$



<u>Another example</u>: In thermal waves, the amplitude of the fundamental fourier component is

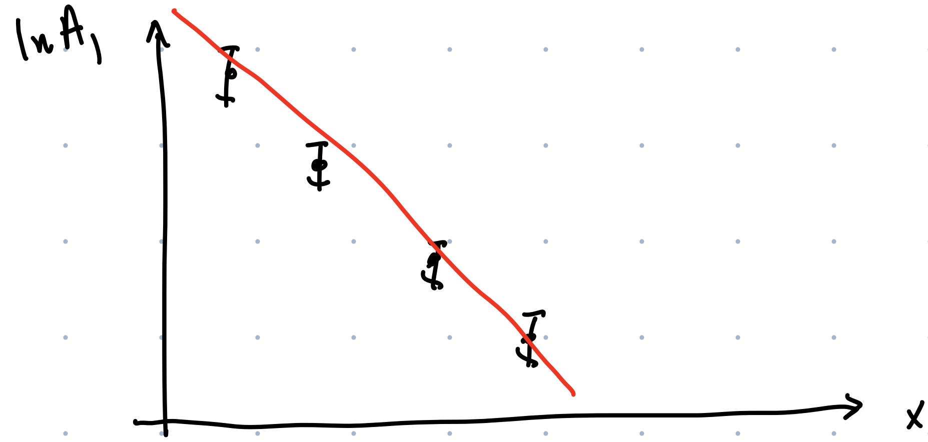
$$A_1 = \left(\frac{4T_0}{\pi}\right)\exp\left(-\sqrt{\frac{\omega}{2\alpha}}\,x\right)$$

To linearize, take the ln of both sides

$$\underbrace{\ln A_1}_{y} = \underbrace{\ln\left(\frac{4T_0}{\pi}\right)}_{a} + \underbrace{\left(-\sqrt{\frac{\omega}{2\alpha}}\right)}_{b}\underbrace{x}_{x}$$

slope b is related to the thermal diffusivity α.

$$b^2 = \frac{\omega}{2\alpha} \Rightarrow \boxed{\alpha = \frac{\omega}{2b^2}}$$



Another example (used in current research)

Low-temperature thermal conductivity of a conductor is typically of the form:

$$K = \underbrace{C_1 T}_{\text{electron contribution.}} + \underbrace{C_3 T^3}_{\text{lattice/phonon contribution}}$$

Meas. K as a fcn of T. Does not appear to be easily linearized at first glance.

Consider

$$\underbrace{\frac{K}{T}}_{\substack{\downarrow \\ y}} = \underset{\substack{\downarrow \\ a}}{C_1} + \underset{\substack{\downarrow \\ b}}{C_3} \underbrace{T^2}_{\substack{\downarrow \\ x}}$$

Plot $\frac{K}{T}$ vs T^2, get intercept $\boxed{a = C_1}$

slope $\boxed{b = C_3}$

Another example: Sometimes a pair of variables cannot be strictly linearized.

In Blackbody radiation

$$I = \frac{2c^2h}{\lambda^5}\left[\frac{1}{e^{hc/\lambda k_B T} - 1}\right]$$

meas I vs T, its <u>not</u> linear.

However, under certain conditions, can make valid approx. that allow the data to be linearized.

In this case, if $\frac{hc}{\lambda k_B T} \gg 1$

then $e^{hc/\lambda k_B T} - 1 \approx e^{hc/\lambda k_B T}$

$$\therefore I \approx \frac{2c^2h}{\lambda^5} e^{-hc/\lambda k_B T}$$

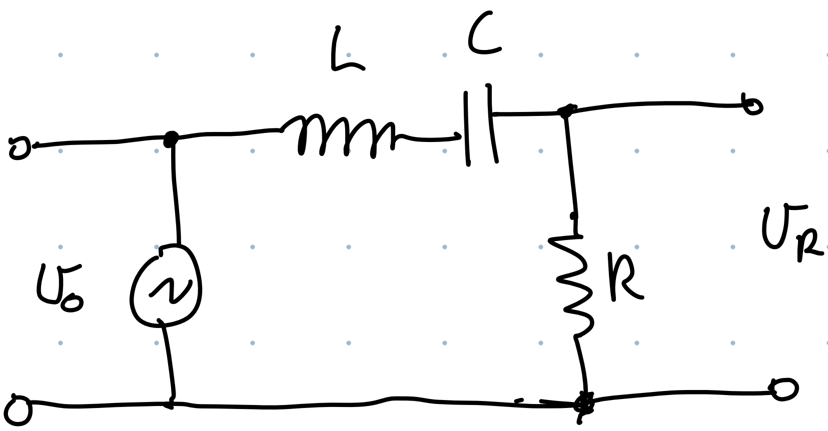
Now, we can take the ln of both sides:

$$\underbrace{\ln I}_{y} = \underbrace{\ln\left(\frac{2c^2h}{\lambda^5}\right)}_{a} + \underbrace{\left(\frac{-hc}{\lambda k_B}\right)}_{b} \underbrace{\frac{1}{T}}_{x}$$

$$\Rightarrow b = -\frac{hc}{\lambda k_B} \quad \therefore \quad \boxed{h = -\frac{\lambda k_B b}{c}}$$

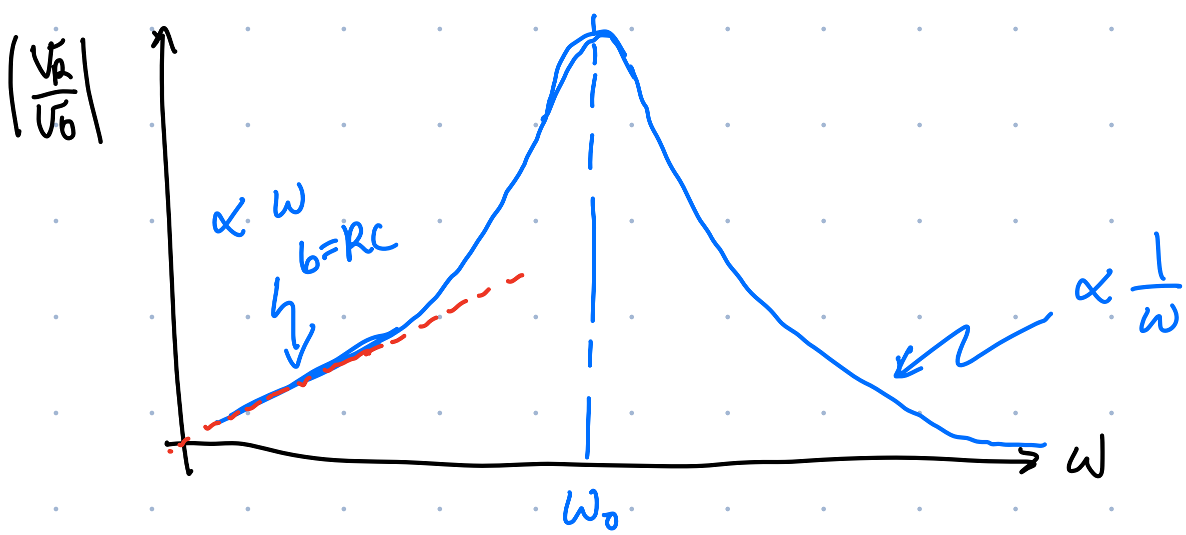
Sometimes it's just not possible to linearize a data set.

Eg. LRC resonance



meas. $\left|\frac{V_R}{V_0}\right|$ vs ω

$$\left|\frac{V_R}{V_0}\right| = \frac{1}{\sqrt{1+\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}$$



$\left|\frac{V_R}{V_0}\right|$ is clearly not linear.

Check behaviour at low freq ($\omega \to 0$)

$$\left|\frac{V_R}{V_0}\right| \approx \frac{1}{\sqrt{1+\left(-\frac{1}{\omega RC}\right)^2}} \approx \frac{1}{\frac{1}{\omega RC}} = \omega RC$$

At low freq. $\left|\frac{V_R}{V_0}\right|$ vs ω is linear w/ slope $b = RC$

Check behaviour at high freq ($\omega \to \infty$)

$$\left|\frac{V_R}{V_0}\right| \approx \frac{1}{\sqrt{1+\left(\omega\frac{L}{R}\right)^2}} \approx \frac{1}{\omega\frac{L}{R}} = \left(\frac{R}{L}\right)\frac{1}{\omega}$$

At high freq, can plot $\left|\frac{V_R}{V_0}\right|$ vs $\frac{1}{\omega}$ to find a slope $b = \frac{R}{L}$

Can we fit the entire freq. dependence?

start w/
$$\left|\frac{V_R}{V_0}\right| = \frac{1}{\sqrt{1+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2}}$$

square both sides...
$$\left|\frac{V_R}{V_0}\right|^2 = \frac{1}{1+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2}$$

invert
$$\left|\frac{V_0}{V_R}\right|^2 = 1+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2$$

expand the square
$$\left|\frac{V_0}{V_R}\right|^2 = \left(1-\frac{2L}{R^2C}\right)+\omega^2\left(\frac{L}{R}\right)^2+\frac{1}{\omega^2}\frac{1}{(RC)^2}$$

<u>Not</u> linearizable

Need some new fitting techniques for fcns that cannot be linearized.

Eg. Calibration of a thermocouple over a large span of temperatures follows a polynomial dependence.

Fitting T vs V to a polynomial

$$T = a_0 + a_1 V + a_2 V^2$$ ← not linearizable.

In general, we want to be able to fit data to a fcn of the form:

$$y(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$$

In fact, the technique that we'll develop can be further generalized to fit data to a fcn of the form:

$$y(x) = a_1 f_1(x) + a_2 f_2(x) + \dots + a_m f_m(x)$$

Here, $f_k(x)$ are fcns of x that do <u>not</u> involve the unknown parameters $a_1, a_2, \dots, a_m$ that we're trying to determine.

For example, for a polynomial up to x^2

$$f_1 = 1 \, , \quad f_2 = x \, , \quad f_3 = x^2$$

s.t. $\quad y = a_1 + a_2 x + a_3 x^2 \rightarrow$ thermocouple example.

or, for the LRC example, we can take

$$f_1 = 1 \, , \quad f_2 = w^2 \, , \quad f_3 = \frac{1}{w^2}$$

st.

$$y = \underbrace{a_1}_{\left(1 - \frac{2L}{R^2 C}\right)} + \underbrace{a_2}_{\left(\frac{L}{R}\right)^2} w^2 + \underbrace{a_3}_{\frac{1}{(RC)^2}} \frac{1}{w^2}$$

Next time, apply method of maximum likelihood to determine the best-fit parameters for fits to fcns of the form:

$$y = a_1 f_1(x) + a_2 f_2(x) + \dots + a_m f_m(x)$$