PHYS 232 March 12, 2025

- Sign up for Experiment #5 by 23:59 $\boxed{\text{today}}$
- Start working on your formal report
  - due April 7 @ 14:00 in SCI 241

Using the method of maximum likelihood, showed that the best-fit slope & intercept for a line through a linear dataset $(x_i, y_i \pm \sigma_i)$ for $i = 1..N$ are given by:

$$\boxed{\begin{aligned} a &= \frac{1}{\Delta}\left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2}\right) \\ \Delta &= \sum \frac{1}{\sigma_i^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2 \end{aligned}}$$

intercept

Likewise:

$$\boxed{b = \frac{1}{\Delta}\left(\sum \frac{1}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2}\right)}$$

slope.

Today: Apply prop. of errors to find σ_a & σ_b

$\Rightarrow$ the uncertainties in the intercept & slope.

Recall: $$\frac{\partial y_i}{\partial y_j} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and $$\sum_{i=1}^{N} f(x_i)\,\delta_{ij} = f(x_j)$$

$$= f(x_1)\cancelto{0}{\delta_{1j}} + f(x_2)\cancelto{0}{\delta_{2j}} + \ldots + \underbrace{f(x_j)\cancelto{1}{\delta_{jj}}}_{i=j \text{ term}} + \ldots + f(x_N)\cancelto{0}{\delta_{Nj}}$$

$$= f(x_j)$$

What are the uncertainties in a & b?

$$a = a(y_1, y_2, \ldots, y_N)$$
$$b = b(y_1, y_2, \ldots, y_N)$$

$$\sigma_a^2 = \sum_{j=1}^{N} \left(\frac{\partial a}{\partial y_j} \sigma_j \right)^2 \qquad (1)$$

$$\sigma_b^2 = \sum_{j=1}^{N} \left(\frac{\partial b}{\partial y_j} \sigma_j \right)^2$$

First, note that Δ is indep. of y_i

Start by evaluating:

$$\frac{\partial a}{\partial y_j} = \frac{1}{\Delta} \frac{\partial}{\partial y_j} \left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} \right)$$

$$= \frac{1}{\Delta} \left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{1}{\sigma_i^2} \underbrace{\frac{\partial y_i}{\partial y_j}}_{\delta_{ij}} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i}{\sigma_i^2} \underbrace{\frac{\partial y_i}{\partial y_j}}_{\delta_{ij}} \right)$$

$$= \frac{1}{\Delta} \left(\sum \frac{x_i^2}{\sigma_i^2} \underbrace{\sum \frac{1}{\sigma_i^2} \delta_{ij}}_{\frac{1}{\sigma_j^2}} - \sum \frac{x_i}{\sigma_i^2} \underbrace{\sum \frac{x_i}{\sigma_i^2} \delta_{ij}}_{\frac{x_j}{\sigma_j^2}} \right)$$

$$\frac{\partial a}{\partial y_j} = \frac{1}{\Delta}\left(\frac{1}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j^2} \sum \frac{x_i}{\sigma_i^2} \right)$$

Sub $\frac{\partial a}{\partial y_j}$ back into ①:

$$\sigma_a^2 = \sum_{j=1}^{N} \left(\frac{\partial a}{\partial y_j} \sigma_j \right)^2$$

$$= \frac{1}{\Delta^2} \sum_{j=1}^{N} \left(\frac{1}{\sigma_j} \sum_i \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j} \sum_i \frac{x_i}{\sigma_i^2} \right)^2$$

Expand the square:

$$\sigma_a^2 = \frac{1}{\Delta^2} \sum_{j=1}^{N} \left[\frac{1}{\sigma_j^2} \left(\sum_i \frac{x_i^2}{\sigma_i^2} \right)^2 + \frac{x_j^2}{\sigma_j^2} \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2 - 2 \frac{x_j}{\sigma_j^2} \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2} \right]$$

Now, distribute the sum over j.

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 + \sum \frac{x_j^2}{\sigma_j^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2 - 2\sum_{j=1}^{N} \frac{x_j}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}\right]$$

$$\left(\sum \frac{x_i}{\sigma_i^2}\right)^2$$

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 + \cancel{\sum \frac{x_j^2}{\sigma_j^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2} - \cancel{2}\sum \frac{x_j^2}{\sigma_j^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2\right]$$

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 - \sum \frac{x_i^2}{\sigma_i^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2\right]$$

common factor

$$\sigma_a^2 = \frac{1}{\Delta^{\cancel{2}}} \sum \frac{x_i^2}{\sigma_i^2} \underbrace{\cancel{\left[\sum \frac{1}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2\right]}}_{\Delta}$$

$$\boxed{\therefore \quad \sigma_a^2 = \frac{1}{\Delta} \sum \frac{x_i^2}{\sigma_i^2}}$$

Uncertainty in the best-fit value for the intercept.

Using the same methods, find uncertainty in the slope is given by:

$$\sigma_b^2 = \sum_{j=1}^{N} \left(\frac{\partial b}{\partial y_j} \sigma_j \right)^2$$

$$\Downarrow$$

$$\boxed{\sigma_b^2 = \frac{1}{\Delta} \sum \frac{1}{\sigma_i^2}}$$

uncertainty in the best-fit value of the slope.

Last time we had:

$$\underbrace{\sum \frac{y_i}{\sigma_i^2}}_{u} = a \underbrace{\sum \frac{1}{\sigma_i^2}}_{u_a} + b \underbrace{\sum \frac{x_i}{\sigma_i^2}}_{u_b}$$

$$\underbrace{\sum \frac{x_i y_i}{\sigma_i^2}}_{v} = a \underbrace{\sum \frac{x_i}{\sigma_i^2}}_{v_a} + b \underbrace{\sum \frac{x_i^2}{\sigma_i^2}}_{v_b}$$

solve system of two eq'ns for 2 unknowns a & b.
Of the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} u_a & u_b \\ v_a & v_b \end{pmatrix}}_{\equiv A} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = A^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\frac{1}{\det A}\begin{pmatrix} V_b & -U_b \\ -V_a & U_b \end{pmatrix}}_{A^{-1}} \begin{pmatrix} a \\ b \end{pmatrix}$$

Recall, we defined $\det A$ to be Δ.

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\frac{1}{\Delta}\begin{pmatrix} V_b & -U_b \\ -V_a & U_a \end{pmatrix}}_{\text{called the covariance matrix.}} \begin{pmatrix} u \\ v \end{pmatrix}$$

For linear fits, the covariance matrix is given by:

$$\frac{1}{\Delta}\begin{pmatrix} \sum \frac{x_i^2}{\sigma_i^2} & -\sum \frac{x_i}{\sigma_i^2} \\ -\sum \frac{x_i}{\sigma_i^2} & \sum \frac{1}{\sigma_i^2} \end{pmatrix}$$

The diagonal elements of the covariance matrix give the squares of the uncertainties in the best-fit parameters.

The 11 element gives $\sigma_a^2 = \frac{1}{\Delta} \sum \frac{x_i^2}{\sigma_i^2}$

The 22 element gives $\sigma_b^2 = \frac{1}{\Delta} \sum \frac{1}{\sigma_i^2}$

Linearizing Experimental Data:

If we can analyze data using linear fits, its good to do b/c we can exactly determine

$a \pm \sigma_a$ (intercept)

$b \pm \sigma_b$ (slope)

This process is easy when a set of meas. is obviously linear.

Eg. Object in free fall

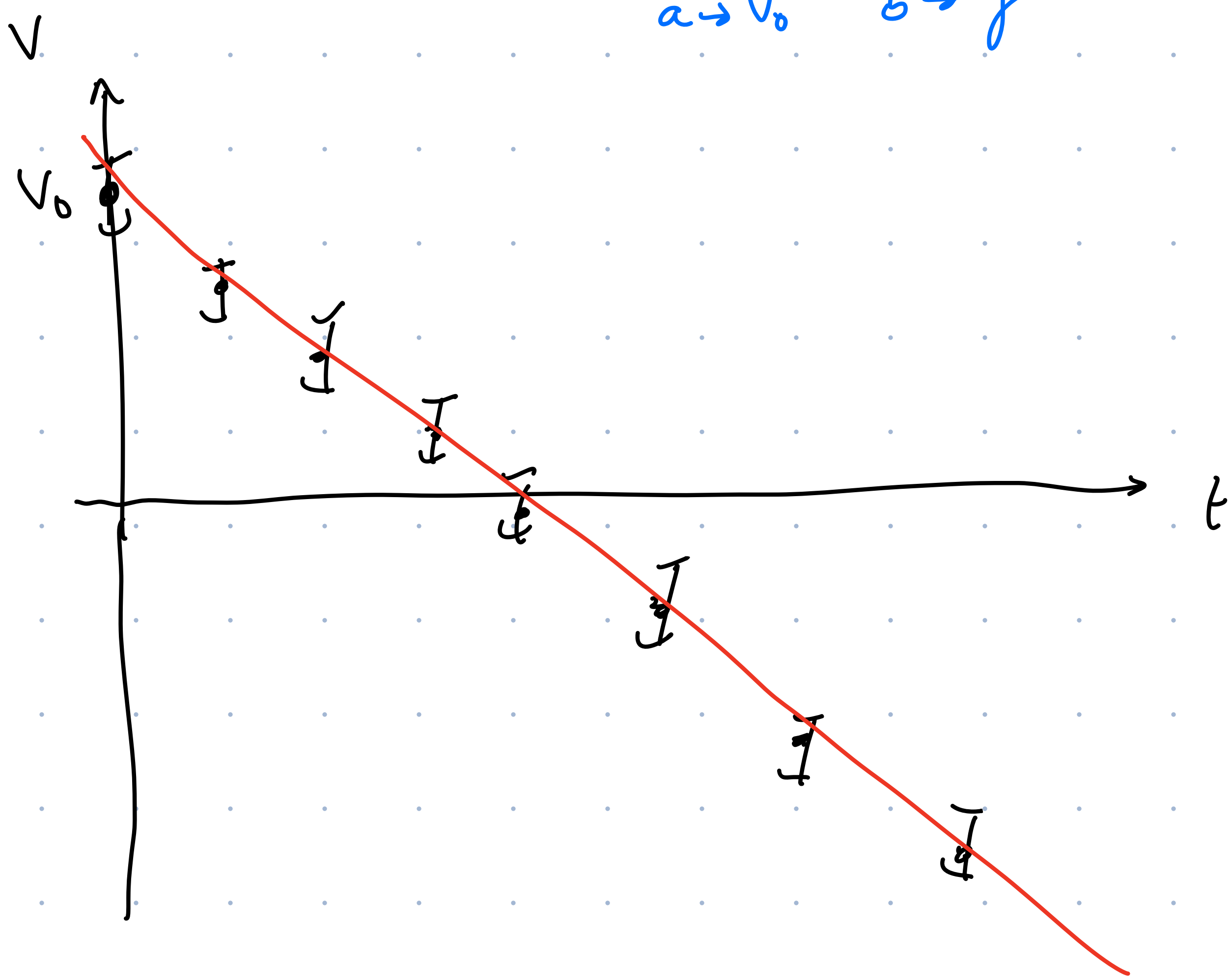
$$V = V_0 - gt$$

of the form

$$y = a + bx$$

$y \to V$ $x \to t$

$a \to V_0$ $b \to -g$



Sometimes, a pair of variable may not be strictly linearly related, but often can still use linear fits to analyze the data.