PHYS 232 March 5, 2025

Assignment #4 on course website
$\rightarrow$ due Wed. March 12 @ 09:00

Last Time: Propagation of Errors

If $y = f(u, v, \ldots)$ & know $u_i \pm \sigma_u$
$v_i \pm \sigma_v$
$\vdots$

Then
$$\boxed{\sigma_y^2 = \sigma_u^2 \left(\left. \frac{\partial f}{\partial u} \right|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\left. \frac{\partial f}{\partial v} \right|_{\bar{v}} \right)^2 + \ldots}$$

<u>Method of Maximum Likelihood</u>

$$P_i = \frac{dx}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{x_i - \mu'}{\sigma} \right)^2 \right]$$

$$P(\mu') = \prod_{i=1}^{N} P_i(\mu')$$

Maximize $P(\mu')$ w.r.t. μ' to find:

$$\mu' = \frac{1}{N}\sum_{i=1}^{N} x_i \rightarrow$$ best estimate of the true value of a quantity based on a set of N identical measurements

Today: Start by finding the uncertainty in μ' using propagation of errors.

Note that μ' is a fcn of $x_1, x_2, \ldots, x_N$

$$\mu'(x_1, x_2, \ldots, x_N)$$

Apply prop. of errors ∴

$$\sigma_{\mu'}^2 = \left(\frac{\partial \mu'}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu'}{\partial x_2}\sigma_2\right)^2 + \ldots + \left(\frac{\partial \mu'}{\partial x_N}\sigma_N\right)^2$$

If we repeat the same meas. method over & over, expect $\sigma_1 = \sigma_2 = \ldots = \sigma_N \equiv \sigma$

$$\sigma_{\mu'}^2 = \sigma^2\left[\left(\frac{\partial\mu'}{\partial x_1}\right)^2 + \left(\frac{\partial\mu'}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial\mu'}{\partial x_N}\right)^2\right]$$

Consider $\displaystyle \frac{\partial\mu'}{\partial x_1} = \frac{\partial}{\partial x_1}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right]$

$$= \frac{1}{N}\frac{\partial}{\partial x_1}\left(x_1 + x_2 + \dots + x_N\right)$$

$$= \frac{1}{N}\underbrace{\frac{\partial x_1}{\partial x_1}}_{1} = \frac{1}{N}$$

Likewise: $\displaystyle \frac{\partial\mu'}{\partial x_2} = \frac{1}{N}, \dots \frac{\partial\mu'}{\partial x_i} = \frac{1}{N} \quad \forall\ i = 1..N$

$$\therefore \sigma_{\mu'}^2 = \sigma^2\left[\underbrace{\left(\frac{1}{N}\right)^2 + \left(\frac{1}{N}\right)^2 + \dots + \left(\frac{1}{N}\right)^2}_{N \text{ terms}}\right]$$

$$= \sigma^2\left[N\left(\frac{1}{N^2}\right)\right] = \frac{\sigma^2}{N}$$

$$\boxed{\therefore \quad \sigma_{\mu'} = \frac{\sigma}{\sqrt{N}} \qquad \text{Standard error or the error in the mean.}}$$

Weighted Mean

How should we combine diff. meas. $x_1, x_2, \ldots, x_N$ when each meas. has a different uncertainty?

$$\sigma_1, \sigma_2, \ldots, \sigma_N$$

Eg. In PHYS 232 lab, can meas. k_B at three diff. temperatures and find

$$k_{B_1} \pm \sigma_1, \quad k_{B_2} \pm \sigma_2, \quad k_{B_3} + \sigma_3$$

Expect meas. w/ small uncertainties to be more important when determining the average.

Again, approach is to use method of maximum likelihood $\Rightarrow$ maximize $P(\mu')$

$$x_1 \pm \sigma_1 \quad , \quad x_2 \pm \sigma_2 \quad , \ldots , \quad x_N \pm \sigma_N$$

$$P_1 = \frac{dx}{\sigma_1 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_1 - \mu'}{\sigma_1}\right)^2\right]$$

$$\vdots$$

$$P(\mu') = \prod_{i=1}^{N} P_i$$

$$= \underbrace{\left[\prod_{i=1}^{N}\left(\frac{dx}{\sqrt{2\pi}\,\sigma_i}\right)\right]}_{\substack{\text{const. w.r.t. } \mu' \\ (\text{indep. of } \mu')}} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i - \mu'}{\sigma_i}\right)^2\right]$$

To maximize $P(\mu')$, minimize

$$X = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i - \mu'}{\sigma_i}\right)^2 \quad \text{w.r.t. } \mu'$$

$$\frac{\partial \chi}{\partial \mu'} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial \mu'} \left(\frac{x_i - \mu'}{\sigma_i} \right)^2 = 0$$

$$= \cancel{\frac{1}{2}} \left[\cancel{2} \sum_{i=1}^{N} \left(\frac{x_i - \mu'}{\sigma_i} \right) \left(-\frac{1}{\sigma_i} \right) \right] = 0$$

$$= -\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2} + \mu' \sum_{i=1}^{N} \frac{1}{\sigma_i^2} = 0$$

Solve for μ':

$$\mu' \sum_{i=1}^{N} \frac{1}{\sigma_i^2} = \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}$$

Finally:

$$\boxed{\mu' = \frac{\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}}$$

Weighted mean.

If all σ_i are equal: $\sigma_i = \sigma$

$$\mu' = \frac{\sum_{i=1}^{N} \frac{x_i}{\sigma^2}}{\sum_{i=1}^{N} \frac{1}{\sigma^2}} = \frac{\cancel{\frac{1}{\sigma^2}} \sum_{i=1}^{N} x_i}{\cancel{\frac{1}{\sigma^2}} \underbrace{\sum_{i=1}^{N} 1}_{N}}$$

$$\therefore \mu' = \frac{1}{N} \sum_{i=1}^{N} x_i \text{ , as expected } \checkmark$$

Silly example:

x_1 :	$1 \pm \sigma$
x_2 :	$1 \pm \sigma$
x_3 :	$1 \pm \sigma$
x_4 :	$2 \pm 2\sigma$
x_5 :	$2 \pm 2\sigma$
x_6 :	$2 \pm 2\sigma$

Here, the regular mean $\frac{1}{N} \sum_i x_i$ gives 1.5.

Expect weighted mean the be less than 1.5 b/c the meas. of $x=1$ have smaller uncertainty.

$$\mu' = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} = \frac{\frac{1}{\sigma^2}+\frac{1}{\sigma^2}+\frac{1}{\sigma^2}+\frac{2}{4\sigma^2}+\frac{2}{4\sigma^2}+\frac{2}{4\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{\sigma^2}+\frac{1}{\sigma^2}+\frac{1}{4\sigma^2}+\frac{1}{4\sigma^2}+\frac{1}{4\sigma^2}}$$

$$= \frac{3(1)+3\left(\frac{2}{4}\right)}{3(1)+3\left(\frac{1}{4}\right)} = \frac{3+\frac{3}{2}}{3+\frac{3}{4}}$$

$$= \frac{\frac{9}{2}}{\frac{15}{4}} = \frac{9}{2}\frac{4}{15} = 2\left(\frac{3}{5}\right)$$

$$= \frac{6}{5} = 1.2$$

As expected, the weighted mean gives a value closer to the meas. w/ small uncertainty.

To find the uncertainty in the weighted mean, apply prop. of errors.

$$\mu' = \frac{\sum_i \frac{x_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}} \qquad \mu'(x_1, x_2, \dots, x_N)$$

$$\sigma_{\mu'}^2 = \left(\frac{\partial \mu'}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu'}{\partial x_2}\sigma_2\right)^2 + \dots + \left(\frac{\partial \mu'}{\partial x_N}\sigma_N\right)^2 \quad \#$$

Consider $\frac{\partial}{\partial x_j}(\mu')$ where $j = 1..N$

$$\frac{\partial \mu'}{\partial x_j} = \frac{\sum_i \frac{\partial}{\partial x_j}\left(\frac{x_i}{\sigma_i^2}\right)}{\sum_i \frac{1}{\sigma_i^2}} = \frac{\sum_i \frac{1}{\sigma_i^2}\frac{\partial x_i}{\partial x_j}}{\sum_i \frac{1}{\sigma_i^2}} \quad *$$

Consider $\dfrac{\partial x_i}{\partial x_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Kroenker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Return to $\circledast$

Kronecker Delta

$$\frac{\partial \mu'}{\partial x_j} = \frac{\sum_i \frac{1}{\sigma_i^2} \frac{\partial x_i}{\partial x_j}}{\sum_i \frac{1}{\sigma_i^2}} = \frac{\sum_i \frac{1}{\sigma_i^2} \delta_{ij}}{\sum_i \frac{1}{\sigma_i^2}}$$

We will use δ_{ij} as a killer of sums.

Eg. consider

$$\sum_i \frac{1}{\sigma_i^2} \delta_{ij} = \frac{1}{\sigma_1^2} \cancelto{0}{\delta_{1j}} + \frac{1}{\sigma_2^2} \cancelto{0}{\delta_{2j}} + \dots + \frac{1}{\sigma_j^2} \cancelto{1}{\delta_{jj}} + \dots + \frac{1}{\sigma_N^2} \cancelto{0}{\delta_{Nj}}$$

$$= \frac{1}{\sigma_j^2}$$

$$\therefore \frac{\partial \mu'}{\partial x_j} = \frac{1/\sigma_j^2}{\sum_i \frac{1}{\sigma_i^2}}$$

Now return to (#) to find:

$$\sigma_{\mu'}^2 = \left(\frac{\frac{1}{\sigma_1^{\cancel{2}}}}{\sum_i \frac{1}{\sigma_i^2}} \cancel{\sigma_1} \right)^2 + \left(\frac{\frac{1}{\sigma_2^{\cancel{2}}}}{\sum_i \frac{1}{\sigma_i^2}} \cancel{\sigma_2} \right)^2 + \dots + \left(\frac{\frac{1}{\sigma_N^{\cancel{2}}}}{\sum_i \frac{1}{\sigma_i^2}} \cancel{\sigma_N} \right)^2$$

$$= \frac{1}{\left(\sum_i \frac{1}{\sigma_i^2} \right)^{\cancel{2}}} \underbrace{\cancel{\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \dots + \frac{1}{\sigma_N^2} \right]}}_{\sum_i \frac{1}{\sigma_i^2}}$$

$$\boxed{\therefore \sigma_{\mu'}^2 = \frac{1}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}}$$

uncertainty in weighted mean.