PHYS 232 Feb. 28, 2024

Last Time: Propagation of Errors

- Fcn of 1 variable

$$y = f(u)$$

If know $u \pm \sigma_u$, then:

$$\sigma_y = \sigma_u \left| \frac{df}{du}\Big|_{\bar{u}} \right|$$

For a fcn of multiple variables

$$y = f(u, v, \ldots)$$

If know $u \pm \sigma_u$, $v \pm \sigma_v$, ...

variance of u

variance v

$$\sigma_y^2 = \sigma_u^2 \left(\frac{\partial f}{\partial u}\Big|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\frac{\partial f}{\partial v}\Big|_{\bar{v}} \right)^2 + \ldots$$

covariance

$$+ 2\sigma_{uv}^2 \left(\frac{\partial f}{\partial u}\Big|_{\bar{u}} \right) \left(\frac{\partial f}{\partial v}\Big|_{\bar{v}} \right) + \ldots$$

$$\sigma_{uv}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (u_i - \bar{u})(v_i - \bar{v})$$

σ_{uv} is called the <u>covariance</u>

If the meas. of u & v follow a Gaussian dist'n,
then

$$\sum_i (u_i - \bar{u}) = 0$$

$$\sum_i (v_i - \bar{v}) = 0$$

$$\sum_i (u_i - \bar{u})(v_i - \bar{v}) = 0$$

$u_i - \bar{u}$	$v_i - \bar{v}$	$(u_i - \bar{u})(v_i - \bar{v})$	
+	+	+	1/4 time
+	−	−	1/4
−	+	−	1/4
−	−	+	1/4

$\hookrightarrow \sum_i (u_i - \bar{u})(v_i - \bar{v}) \approx 0$ for Gaussian-distributed meas.

In this case, covariance $\sigma_{uv}^2 = 0$.

The covariance is zero when u_i & v_i meas. are independent of one another.

In this case, the prop. of errors becomes:

If $y = f(u, v, \ldots)$ & know $u_i \pm \sigma_u$
$v_i \pm \sigma_v$
$\vdots$

Then

$$\boxed{\sigma_y^2 = \sigma_u^2 \left(\left. \frac{\partial f}{\partial u} \right|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\left. \frac{\partial f}{\partial v} \right|_{\bar{v}} \right)^2 + \ldots}$$

Example: Boltzmann's Constant Lab.

$$m = \frac{e}{k_B T}$$

Plot $\ln I$ vs V, get a straight line w/ slope m.

$$\Rightarrow k_B = \frac{e}{mT}$$

know $T \pm \sigma_T$, $m \pm \sigma_m$, $e \pm \sigma_e$

assume that uncertainty in e is small

Find σ_{k_B} → Apply prop. of errors.

$$\sigma_{k_B}^2 = \sigma_T^2 \left(\frac{\partial k_B}{\partial T}\right)^2 + \sigma_m^2 \left(\frac{\partial k_B}{\partial m}\right)^2 + \sigma_e^2 \left(\frac{\partial k_B}{\partial e}\right)^2$$

$$\frac{\partial k_B}{\partial T} = \frac{\partial}{\partial T}\left(\frac{e}{mT}\right) = -\frac{e}{m}\frac{1}{T^2}$$

$$\frac{\partial k_B}{m} = -\frac{e}{m^2 T}$$

$$\frac{\partial k_B}{\partial e} = \frac{1}{mT}$$

$$\therefore \sigma_{k_B}^2 = \left(\sigma_T \frac{e}{mT^2}\right)^2 + \left(\sigma_m \frac{e}{m^2T}\right)^2 + \left(\sigma_e \frac{1}{mT}\right)^2$$

All quantities on RHS known. Sub in values to find σ_{k_B}.

$$\left(\frac{\sigma_{k_B}}{k_B}\right)^2 = \left(\frac{\sigma_T \frac{\cancel{e}}{\cancel{m}T^{\cancel{2}}}}{\frac{\cancel{e}}{\cancel{m}\cancel{T}}}\right)^2 + \left(\frac{\sigma_m \frac{\cancel{e}}{m^{\cancel{2}}\cancel{T}}}{\frac{\cancel{e}}{\cancel{m}\cancel{T}}}\right)^2 + \left(\frac{\sigma_e \frac{1}{\cancel{m}\cancel{T}}}{\frac{e}{\cancel{m}\cancel{T}}}\right)^2$$

$$\therefore \left(\frac{\sigma_{k_B}}{k_B}\right)^2 = \left(\frac{\sigma_T}{T}\right)^2 + \left(\frac{\sigma_m}{m}\right)^2 + \left(\frac{\sigma_e}{e}\right)^2$$

$$\therefore \frac{\sigma_{k_B}}{k_B} = \sqrt{\left(\frac{\sigma_T}{T}\right)^2 + \left(\frac{\sigma_m}{m}\right)^2 + \left(\frac{\sigma_e}{e}\right)^2}$$

Notice that this result is similar to the first-year method of adding fractional errors. The only difference is $(\ )^2$ & $\sqrt{\ }$ (quadrature sum) if $\frac{\sigma_e}{e} \ll \frac{\sigma_T}{T}$ & $\frac{\sigma_m}{m}$

then

$$\frac{\sigma_{k_B}}{k_B} \approx \sqrt{\left(\frac{\sigma_m}{m}\right)^2 + \left(\frac{\sigma_T}{T}\right)^2}$$

when a term has small fraction uncertainty w.r.t. all other terms, it can be ignored.

Eg. $y = \ln x$

$$\sigma_y = \sigma_x \left| \frac{dy}{dx} \right| \qquad \frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\boxed{\therefore \sigma_y = \frac{\sigma_x}{x}}$$

Suppose we've meas. a quantity x, N times. What is the best possible estimate of the true value of x?

$$x_1, x_2, x_3, \ldots \; x_N$$

Expect to find that the best estimate of true value is given by the mean $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$

Method of Maximum Likelihood.

If the meas. are Gaussian dist'd

Prob. that meas. i falls between x_i & $x_i + dx$ is :

$$P_i = \frac{dx}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu'}{\sigma}\right)^2\right]$$

Assume we don't know true value of μ a head of time. Write μ as μ' & deduce our best est. for μ.

Total prob. of meas. x_1 in trial 1
x_2 " " 2
$\vdots$
x_N N

is given by:

$$P(\mu') = P_1 P_2 \cdots P_N$$

$$= \prod_{i=1}^{N} P_i(\mu') = \left(\frac{dx}{\sigma\sqrt{2\pi}}\right)^N \cdot$$

$$\exp\left[-\frac{1}{2}\left(\frac{x_1-\mu'}{\sigma}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{x_2-\mu'}{\sigma}\right)^2\right] \cdot \cdots$$

$$\cdot \exp\left[-\frac{1}{2}\left(\frac{x_N-\mu'}{\sigma}\right)^2\right]$$

$$\therefore P(\mu') = \left(\frac{dx}{\sigma\sqrt{2\pi}}\right)^N \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma}\right)^2\right]$$

The method of maximum likelihood assumes that actual dataset & value of μ' give max. value of $P(\mu')$.

Evaluate $$\frac{\partial P(\mu')}{\partial \mu'} = 0$$ & solve for μ'.

Equivalently, we can minimize

$$X \equiv \frac{1}{2} \sum_{i=1}^{N} \left(\frac{x_i - \mu'}{\sigma} \right)^2$$

$$\frac{\partial X}{\partial \mu'} = \frac{\partial}{\partial \mu'} \left[\frac{1}{2} \sum_{i=1}^{N} \left(\frac{x_i - \mu'}{\sigma} \right)^2 \right]$$

$$= \frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial \mu'} \left(\frac{x_i - \mu'}{\sigma} \right)^2 = 0$$

$$= \frac{1}{\cancel{2}} \sum_{i=1}^{N} \cancel{2} \left(\frac{x_i - \mu'}{\sigma} \right) \left(-\frac{1}{\sigma} \right) = 0$$

$$= \sum_{i=1}^{N}\left(-\frac{x_i}{\cancel{\sigma^2}}\right) + \sum_{i=1}^{N} \frac{\mu'}{\cancel{\sigma^2}} = 0$$

$$\therefore \; -\sum_{i=1}^{N} x_i + \mu' \underbrace{\sum_{i=1}^{N} 1}_{N} = 0$$

solve for μ':

$$\mu' N = \sum_{i=1}^{N} x_i$$

$$\boxed{\therefore \; \mu' = \frac{1}{N}\sum_{i=1}^{N} x_i}$$ as expected ✓