PHYS 232 Feb. 26, 2024

Last Time: "Derivation" of Gaussian distribution.

$\Rightarrow$ Found $$P_G = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Prob. of a measurement falling between x_1 & x_2 is:

$$P(x_1, x_2) = \int_{x_1}^{x_2} P_G(x; \mu, \sigma)\, dx$$

68-95-99.7 Rule

Prob. of a measurement falling within one std. dev. of mean is:

$$\mu - \sigma < x < \mu + \sigma \Rightarrow 68\%$$

$$\mu - 2\sigma < x < \mu + 2\sigma \Rightarrow 95\%$$

$$\mu - 3\sigma < x < \mu + 3\sigma \Rightarrow 99.7\%$$

Summary of Prob. dist'ns:

Binomial:

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\mu = np \qquad \sigma^2 = np(1-p)$$

Poisson:

$p \ll 1$ limit of Binomial dist'n

(small prob. of success in any individual trial)

$$P_P(x; \mu) = \frac{\mu^x}{x!} e^{-\mu}$$

mean: μ $\qquad \sigma = \sqrt{\mu}$

Relevant for counting experiments

Gaussian:

n large & $np \gg 1$ limit of Binomial distr'n.

$$P_G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

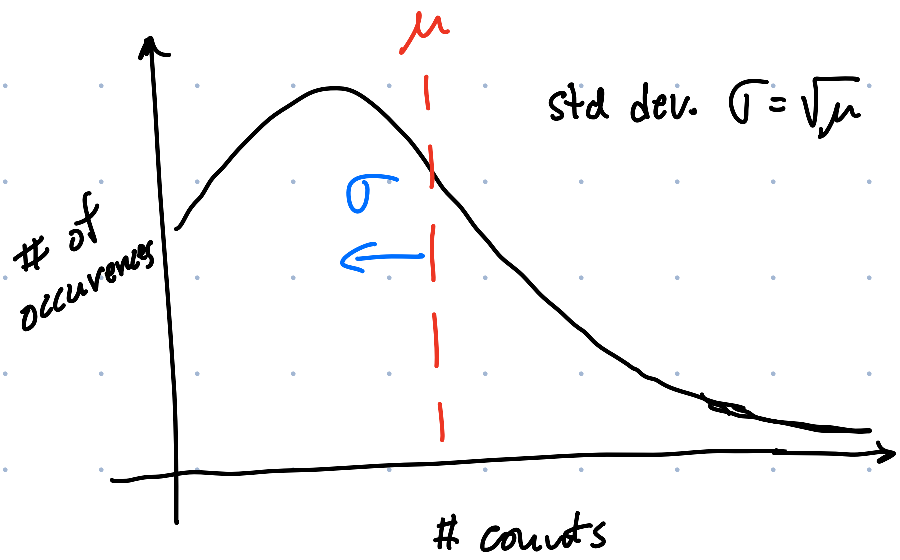
mean: μ std. dev.: σ

Uncertainty in Counting Exp'ts.

In practice can meas. no. of counts in a time interval Δt w/ absolute certainty.

However, if repeat meas. many times will get statistical fluctuations in the no. of counts.

trials	counts
1	18
2	24
3	21
4	19
⋮	⋮



often not possible to conduct repeated experiments to map out the full dist'n in order to properly estimate μ.

∴ in counting exp'ts often do the meas. once & assume that our result (no. of counts) is a reasonable (best possible) estimate of μ.

Assume that $\mu = N \Rightarrow \sigma = \sqrt{N}$

no. of counts — uncertainty in no. of counts.

Notice that the fractional uncertainty

$$\frac{\sigma}{\mu} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$$ shrinks as N increases.

Propagation of Errors

Suppose we want to determine some quantity y that depends on meas. values:

$$u \pm \sigma_u \quad \& \quad v \pm \sigma_v$$

If $y = f(u, v)$, what is σ_y?

Eg. In Boltzmann's const.

$$\ln I \approx \ln I_0 + \left(\frac{e}{k_B T}\right) V \qquad \text{slope}$$

If you plot $\ln I$ vs V, slope is equal to $m = \dfrac{e}{k_B T}$

Assume that you've determined $m \pm \sigma_m$ & $T \pm \sigma_T$.
Now, you want to calc $k_B \pm \sigma_{k_B}$ (assume e is known)

$$k_B = \frac{e}{mT} \qquad \text{now} \quad k_B(m, T)$$

know $m \pm \sigma_m$, $T \pm \sigma_T$ how do we find σ_{k_B}?

Start w/ a fcn of one variable: $y = f(u)$ know $u \pm \sigma_u$

Want to find σ_y.

Can approx $f(u)$ using a Taylor series:

$$f(u) \approx f(\bar{u}) + (u-\bar{u})\, f'(u)\Big|_{u=\bar{u}}$$

assume that $f(\bar{u}) \approx \bar{f}$

$$\underbrace{f(u)}_{f} \approx \bar{f} + (u-\bar{u})\, f'(\bar{u})$$

$$\therefore \quad f - \bar{f} = (u-\bar{u})\, f'(\bar{u})$$

$$\therefore \quad f - \bar{f} = (u-\bar{u})\, \frac{df}{du}\Big|_{\bar{u}}$$

Recall that in general

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

In a similar way, we can find σ_f by:

$$\sigma_f^2 = \frac{1}{N-1}\sum_{i=1}^{N}(\underbrace{f_i}_{f(u_i)} - \bar{f})^2$$

$$= \frac{1}{N-1}\sum_{i=1}^{N} (u_i - \bar{u})^2 \underbrace{\left(\left.\frac{df}{du}\right|_{\bar{u}}\right)^2}_{\text{indep. of } i}$$

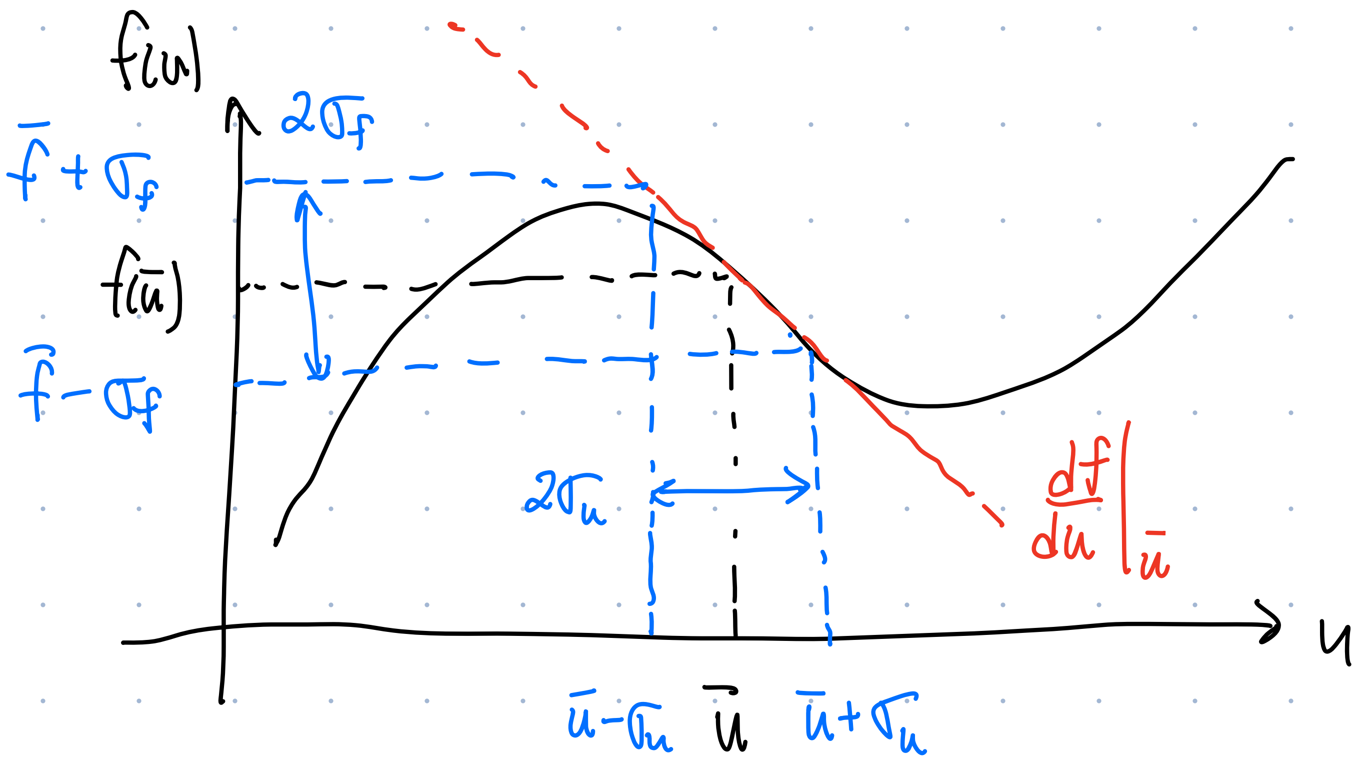
indep. of i
can be taken outside of sum.

$$\sigma_f^2 = \underbrace{\left[\frac{1}{N-1}\sum_{i=1}^{N}(u_i - \bar{u})^2\right]}_{\sigma_u^2}\left(\left.\frac{df}{du}\right|_{\bar{u}}\right)^2$$

$$\therefore \quad \sigma_f^{\,2} = \sigma_u^{\,2} \left(\left. \frac{df}{du} \right|_{\bar{u}} \right)^2$$

Prop. of errors for a fcn of one variable:

$$\boxed{\sigma_f = \sigma_u \left| \left. \frac{df}{du} \right|_{\bar{x}} \right|} \qquad f(u)$$



Slope of the red line is given by:

① $\left. \frac{df}{du} \right|_{\bar{u}}$ ② $\frac{\text{rise}}{\text{run}} = \frac{\cancel{2}\,\sigma_f}{\cancel{2}\,\sigma_u}$

① = ②

$$\therefore \quad \frac{\sigma_f}{\sigma_u} \simeq \left.\frac{df}{du}\right|_{\bar{u}}$$

$$\Rightarrow \quad \boxed{\sigma_f = \sigma_u \left| \left.\frac{df}{du}\right|_{\bar{u}} \right|}$$

ensure that $\sigma_f > 0$.

as expected.

For a fcn of more than one variable

$$y = f(u, v, \ldots)$$

The first order Taylor series expansion is:

$$y = f(u, v, \ldots) \approx f(\bar{u}, \bar{v}, \ldots) + (u - \bar{u}) \left.\frac{\partial f}{\partial u}\right|_{\bar{u}} + (v - \bar{v}) \left.\frac{\partial f}{\partial v}\right|_{\bar{v}} + \ldots$$

Assume $f(\bar{u}, \bar{v}, \ldots) = \bar{f}$

$$f - \bar{f} \approx (u - \bar{u}) \frac{\partial f}{\partial u}\bigg|_{\bar{u}} + (v - \bar{v}) \frac{\partial f}{\partial v}\bigg|_{\bar{v}} + \ldots$$

$$\sigma_f^2 \approx \frac{1}{N-1} \sum_{i=1}^{N} (f_i - \bar{f})^2$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \left[(u_i - \bar{u}) \frac{\partial f}{\partial u}\bigg|_{\bar{u}} + (v_i - \bar{v}) \frac{\partial f}{\partial v}\bigg|_{\bar{v}} + \ldots \right]^2$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \Bigg[\overbrace{(u_i - \bar{u})^2 \left(\frac{\partial f}{\partial u}\bigg|_{\bar{u}} \right)^2}^{\sigma_u^2 \left(\frac{\partial f}{\partial u}\big|_{\bar{u}} \right)^2} + \overbrace{(v_i - \bar{v})^2 \left(\frac{\partial f}{\partial v}\bigg|_{\bar{v}} \right)^2}^{\sigma_v^2 \left(\frac{\partial f}{\partial v}\big|_{\bar{v}} \right)^2}$$

$$+ \ldots + \underbrace{2(u_i - u)(v_i - v) \left(\frac{\partial f}{\partial u}\bigg|_{\bar{u}} \right) \left(\frac{\partial f}{\partial v}\bigg|_{\bar{v}} \right)}_{\equiv 2\sigma_{uv}^2 \left(\frac{\partial f}{\partial u}\big|_{\bar{u}} \right) \left(\frac{\partial f}{\partial v}\big|_{\bar{v}} \right)}$$

$$+ \ldots \Bigg]$$

covariance

$$\sigma_f^2 = \sigma_u^2 \left(\frac{\partial f}{\partial u}\bigg|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\frac{\partial f}{\partial v}\bigg|_{\bar{v}} \right)^2 + \ldots$$

$$+ 2\sigma_{uv}^2 \left(\frac{\partial f}{\partial u}\bigg|_{\bar{u}} \right) \left(\frac{\partial f}{\partial v}\bigg|_{\bar{v}} \right) + \dots$$