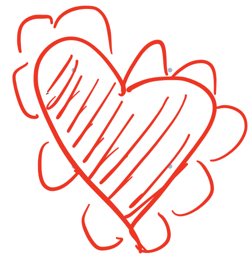
PHYS 232 Feb. 14, 2025

- Assignment #3 on course website. Due Feb. 28



<u>Last Time:</u> "Derivation of a Gaussian"

Find prob. of taking m extra steps to the right when we take a total of n steps & prob. of stepping right/left is $1/2$.

Steps right $\quad x = \frac{n}{2} + \frac{m}{2}$

steps left $\quad n - x = \frac{n}{2} - \frac{m}{2}$

Extra steps right

$$x - (n-x)$$

$$= \left(\frac{n}{2} + \frac{m}{2}\right) - \left(\frac{n}{2} - \frac{m}{2}\right)$$

$$= m$$

$$\mu = np = \frac{n}{2}$$

$$\sigma^2 = np(1-p) = \frac{n}{4}$$

Assume n is large & $m \ll n$.

Starting from the Binomial distribution & making use of Striling's approx. for the factorials, we found:

$$\therefore \; P(m,n) = \boxed{\frac{n^n \sqrt{n}\, 2^{-n}}{\sqrt{2\pi}\sqrt{\frac{n^2}{4}-\frac{m^2}{4}}\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{\frac{n}{2}}\underline{\left(\frac{n+m}{n-m}\right)^{m/2}}}}$$

So far, the only approx. we've made is Striling's approximation. We now make some more...

We will make use of the following

$$(1 \pm x)^p \qquad |x| \ll 1$$

$$\approx 1 \pm px \qquad \text{Binomial}$$

$$\ln(1+x) \qquad |x| \ll 1$$

$$\approx x$$

Start by considering

$$\left(\frac{n+m}{n-m}\right)^{m/2} = \left(\frac{1+\frac{m}{n}}{1-\frac{m}{n}}\right)^{m/2}$$

$$= \left[\left(1+\frac{m}{n}\right)\underbrace{\left(1-\frac{m}{n}\right)^{-1}}_{\approx 1+\frac{m}{n}}\right]^{m/2}$$

binomial approx.

$$= \left\{\left[\left(1+\frac{m}{n}\right)^2\right]^{1/2}\right\}^m = \underline{\left(1+\frac{m}{n}\right)^m}$$

$$\therefore P(m,n) \approx \frac{\boxed{n^n}\,\boxed{\sqrt{n}}\,\boxed{2^{-n}}}{\boxed{\sqrt{2\pi}}\sqrt{\frac{n^2}{4}-\frac{m^2}{4}}\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{n/2}\left(1+\frac{m}{n}\right)^m}$$

$$\approx \frac{\sqrt{\frac{n}{2\pi}}\cancel{\left(\frac{n}{2}\right)^n}}{\frac{n}{2}\sqrt{1-\left(\frac{m}{n}\right)^2}\cancel{\left(\frac{n}{2}\right)^n}\left[1-\left(\frac{m}{n}\right)^2\right]^{n/2}\left(1+\frac{m}{n}\right)^m}$$

$$P(m,n) \approx \frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\left[1-\left(\frac{m}{n}\right)^2\right]^{-1/2}\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{n}{2}}\left(1+\frac{m}{n}\right)^{-m}$$

Now, take the ln of both sides & prepare to use $\ln(1\pm x) \approx \pm x$

$$\ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right)-\frac{1}{2}\ln n-\frac{1}{2}\overbrace{\ln\left[1-\left(\frac{m}{n}\right)^2\right]}^{-\left(\frac{m}{n}\right)^2}$$

$$-\frac{n}{2}\ln\left[1-\left(\frac{m}{n}\right)^2\right]-m\ln\left(1+\frac{m}{n}\right)$$

$$-\left(\frac{m}{n}\right)^2 \qquad\qquad + \frac{m}{n}$$

$$\therefore \ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln n + \cancel{\frac{1}{2}\left(\frac{m}{n}\right)^2} \rightarrow 0$$

$$+ \frac{1}{2}\frac{m^2}{n} - \frac{m^2}{n}$$

ignore b/c small c.t. m^2/n terms

Recall that $n \gg m$ $\therefore \left(\frac{m}{n}\right)^2 \ll \frac{m^2}{n}$

$$\therefore \ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln n - \frac{1}{2}\frac{m^2}{n}$$

$$\approx \ln\left(\frac{2}{\sqrt{2\pi}\sqrt{n}}\right) - \frac{m^2}{2n}$$

Now exponentiate:

$$P(m,n) \approx \frac{2}{\sqrt{2\pi}\sqrt{n}} e^{-m^2/2n}$$

$$\frac{1}{\sigma}$$

Recall $\mu = \frac{n}{2}$ $\quad \sigma^2 = \frac{n}{4} \Rightarrow \sigma = \frac{\sqrt{n}}{2}$

$$P(m\ n) \simeq \frac{1}{\sqrt{2\pi}\,\sigma} e^{-m^2/2n}$$

$$x = \underbrace{\frac{n}{2}}_{\mu} + \frac{m}{2} \qquad \therefore\ m = 2(x-\mu)$$

$$\therefore\ P(m,n) \simeq \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{4(x-\mu)^2}{n}\right]}$$

$$\frac{1}{\sigma^2}$$

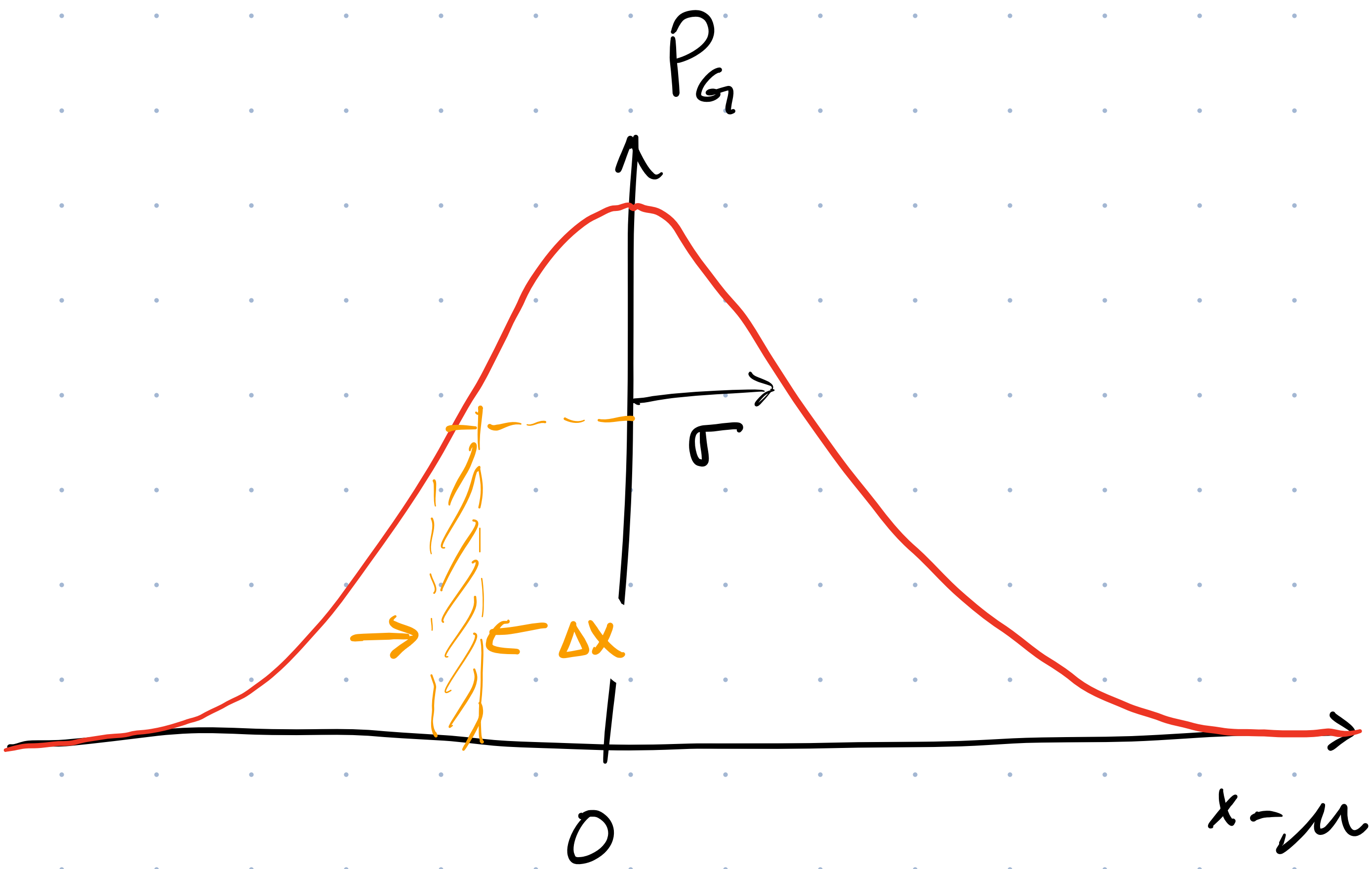
Finally arrive @ the Gaussian dist'n

$$P(m,n) \equiv \boxed{P_G = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

Properties of Gaussian Dist'n.

- symmetric since it depends only on

$$e^{-(x-\mu)^2}$$



Total area under the curve is 1.

$$\int_{x=-\infty}^{+\infty} P_G \, dx = 1$$

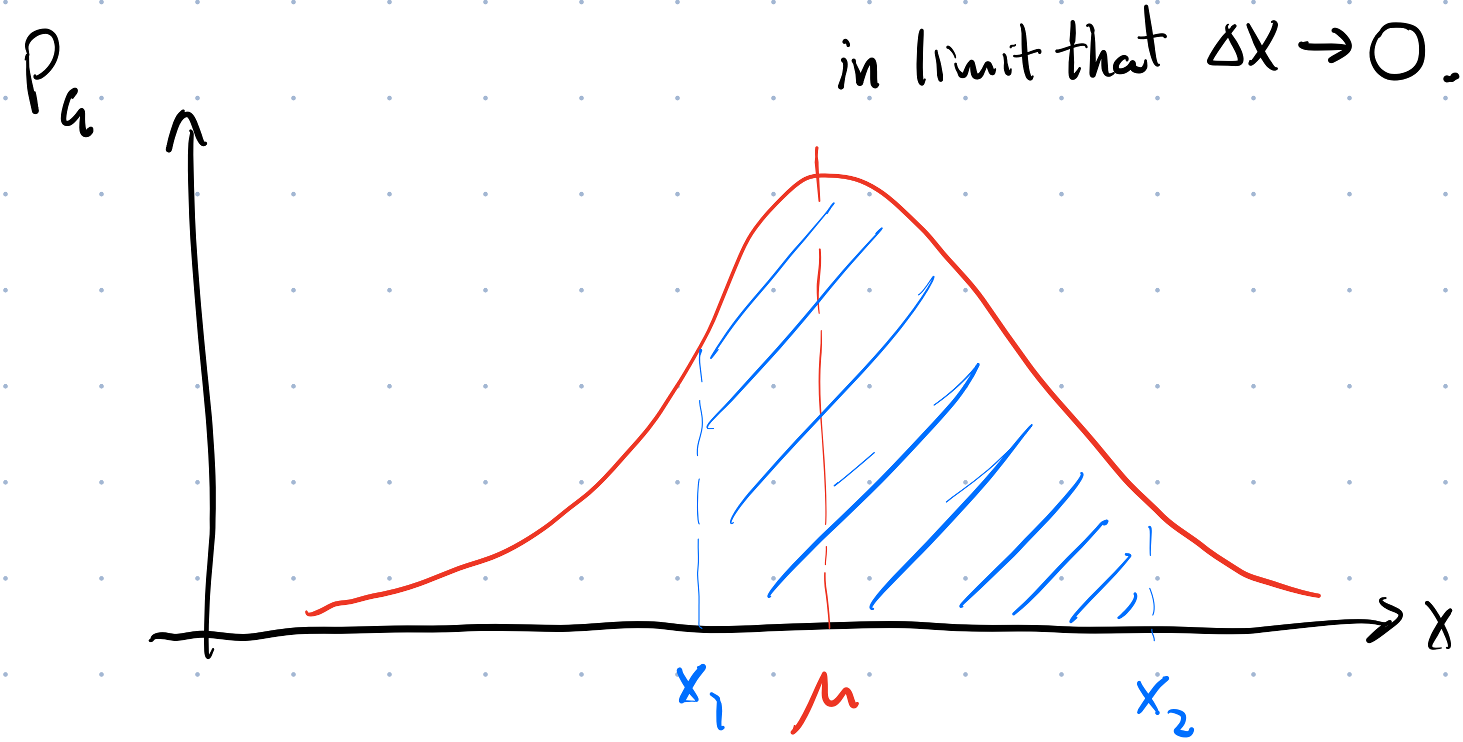
Prob of a meas. falling between x & $x+dx$ is:

$$\Delta P = P_G(x)\,\Delta x$$

Prob. of finding a meas. between x_1 & x_2 is:

$$P(x_1 \to x_2) = \sum \Delta P = \int_{x_1}^{x_2} P_G(x)\,dx$$

in limit that $\Delta x \to 0$.



Eg. Find prob. of a meas. falling within Δx of the mean. i.e. between $\mu - \Delta x$ & $\mu + \Delta x$.

$$P(\mu - \Delta x \to \mu + \Delta x) = \int_{\mu-\Delta x}^{\mu+\Delta x} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

make a substitution

$$z = \frac{x-\mu}{\sigma}$$

$$dz = \frac{dx}{\sigma} \Rightarrow dx = \sigma\, dz$$

when $x = \mu \pm \Delta x$, $z = \dfrac{\cancel{\mu} \pm \Delta x - \cancel{\mu}}{\sigma}$

$$= \pm \frac{\Delta x}{\sigma}$$

$$P(\mu-\Delta x \rightarrow \mu+\Delta x) = \int_{-\frac{\Delta x}{\sigma}}^{\Delta x/\sigma} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \exp\left[-\frac{z^2}{2}\right] \cancel{\sigma}\, dz$$

$$= \boxed{-\frac{1}{\sqrt{2\pi}} \int_{-\frac{\Delta x}{\sigma}}^{\Delta x/\sigma} e^{-z^2/2}\, dz}$$

Cannot evaluate this integral analytically, must use numerical methods.

$\Delta x/\sigma$	$P(\mu-\Delta x \rightarrow \mu+\Delta x)$	
1	0.683	1σ within μ
2	0.954	2σ within μ
3	0.997	⋮
4	0.999937	
5	0.9999994	

68-95-99.7 rule.