PHYS 232 Feb. 12, 2025

- Assignment #3 on course website. Due Feb. 28

<u>Last Time:</u> Poisson Dist'n

$$\lim_{p\to 0} P_B(x;n,p) = \boxed{P_P(x;\mu) = \frac{\mu^x}{x!}e^{-\mu}}$$

$$\sum_{x=0}^{\infty} P_P(x;\mu) = 1$$

$$\bar{x} = \mu$$

$$\sigma = \sqrt{\mu}$$

<u>Example</u>: Avg. no. of flaws in a fibre optic cable of length 50 m is 1.2.

(i) What is the prob. of exactly 3 flaws in 150 m of cable?

$$P_p = \frac{\mu^x}{x!} e^{-\mu}$$

For 150 m of cable, expect avg. no. of flaws to be $\mu = 3(1.2) = 3.6$

If we require $x = 3$ flaws:

$$P_p = \frac{(3.6)^3}{3!} e^{-3.6} = \underline{\underline{0.212}}$$

Note that since P_p is a limit of P_B, expect to get a similar result when using P_B.

$$P_B = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$n = 150$ (each 1m length is like a trial)

$x = 3$ (looking for 3 flaws)

$p = \frac{1.2}{50} = 0.024$ prob. of a flaw in 1m of cable.

$$P_B = (3; 150, 0.024) = \frac{150!}{3!(147)!}\left(\frac{1.2}{50}\right)^3\left(1-\frac{1.2}{50}\right)^{147}$$

$$= \underline{\underline{0.214}}$$ close to P_p as expected!

(ii) what is prob. of <u>at least</u> two flaws in 100m of cable?

Prob. of getting $x = 2, 3, 4, 5, \ldots \infty$ flaws.

$= 1 - $ (prob. of getting 0 or 1 flaw)

$1 - P_p(0, \mu) - P_p(1, \mu)$

1.2 flaws per 50m

for 100m length $\mu = 2(1.2) = 2.4$

$$1 - \underset{P_p(0;\mu)}{0.091} - \underset{P_p(1;\mu)}{0.218} = \boxed{0.693}$$

Using Binomial dist'n.

$p = 0.024$ for 1m length like before.

$$P_B(0;100,p) + P_B(1;100,p)$$

$$= \underbrace{\frac{100!}{0!\,100!} p^0 (1-p)^{100}}_{(1-p)^{100}} + \underbrace{\frac{100!}{1!\,99!} p^1 (1-p)^{99}}_{100p(1-p)^{99}}$$

$$= 0.088 + 0.217 = 0.305$$

$$\therefore 1 - .305 = \boxed{0.695}$$

(iii) What is prob. of exactly one flaw in the first 50m of a cable & exactly one flaw in the next 50m of cable?

$\mu = 1.2$
$x = 1$

$P_p(1, 1.2) = 0.361$ Prob. of 1 flaw in any 50 m section.

$$P = P_p(1; 1.2)\, P_p(1; 1.2) = \boxed{0.131}$$

w/ Binomial dist'n.

$$p = \frac{1.2}{50} = 0.024 \quad n = 50 \quad x = 1$$

$$P = \left[P_B(1; 50, p) \right]^2 = \left[\frac{50!}{1!\,49!} p^1 (1-p)^{49} \right]^2$$

$$= \left[50\,p\,(1-p)^{49} \right]^2 = \boxed{0.133}$$

"Derivation" of Gaussian Dist'n.

We will show that we can start from the binomial dist'n & through some manipulations / approximations can arrive @ Gaussian dist'n.

Gaussian dist'n arises when fluctation in a meas. quantity are random or symmetric about the mean. Binomial dist'n is symmetric for $p = \frac{1}{2}$. $\therefore$ we will start w/ $p = \frac{1}{2}$.

Recall the random walk problem. Prob of taking x of n steps to the right is given by:

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

know $\mu = np$

$\sigma^2 = np(1-p)$

Take $p = \frac{1}{2}$ & consider the prob. of taking

m more steps to right than to left.

For $p = \frac{1}{2}$ $\boxed{\mu = \frac{n}{2} \qquad \sigma^2 = \frac{n}{4}}$

$X = \frac{n}{2} + \frac{m}{2}$ $\qquad$ $\underbrace{n-x}_{\text{no. of steps left.}} = \frac{n}{2} - \frac{m}{2}$

no. of steps right

No. of "extra" steps right is then:

$$X - (n-x) = \left(\cancel{\frac{n}{2}} + \frac{m}{2}\right) - \left(\cancel{\frac{n}{2}} - \frac{m}{2}\right)$$

$= m$ extra steps right. ✓

Prob. of m extra steps right:

$$P(m,n) = \frac{n!}{\underbrace{\left(\frac{n}{2}+\frac{m}{2}\right)!}_{x!}\underbrace{\left(\frac{n}{2}-\frac{m}{2}\right)!}_{(n-x)!}} \underbrace{\left(\frac{1}{2}\right)^{\frac{n}{2}+\frac{m}{2}}\left(\frac{1}{2}\right)^{\frac{n}{2}-\frac{m}{2}}}_{\left(\frac{1}{2}\right)^{n}}$$

$$P(m,n) = \frac{n!}{\left(\frac{n}{2}+\frac{m}{2}\right)!\left(\frac{n}{2}-\frac{m}{2}\right)!} \left(\frac{1}{2}\right)^{n}$$

To make the factorials easier to work w/ mathematically, use Striling's approx.

$$z! \approx \sqrt{2\pi}\, z^{z} e^{-z} \sqrt{z} \quad \text{valid for } z \gg 1$$

$$\text{eg. } 10! = \underbrace{3.6288 \times 10^{6}}_{\text{exact}} \approx \underbrace{3.5987 \times 10^{6}}_{\text{stirling's approx}}$$

diff of 0.83%

Assume that $m \ll n$ s.t. all factorials in $P(m,n)$ are large nos.

$$P(m,n) = \frac{\cancel{\sqrt{2\pi}}\, n^{n}\, \cancel{e^{-n}}\, \sqrt{n}\; \overset{\left(\frac{1}{2}\right)^{n}}{2^{-n}}}{\left[\left(\cancel{\sqrt{2\pi}} \left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{n}{2}+\frac{m}{2}} \cancel{e}^{\,-\cancel{\frac{n}{2}}-\cancel{\frac{m}{2}}} \sqrt{\frac{n}{2}+\frac{m}{2}}\right) \cdot \left(\sqrt{2\pi} \left(\frac{n}{2}-\frac{m}{2}\right)^{\frac{n}{2}-\frac{m}{2}} \cancel{e}^{\,-\cancel{\frac{n}{2}}+\cancel{\frac{m}{2}}} \sqrt{\frac{n}{2}-\frac{m}{2}}\right)\right]}$$

$$P(m,n) = \frac{n^n \sqrt{n}\, 2^{-n}}{\sqrt{2\pi}\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{n}{2}+\frac{m}{2}} \sqrt{\frac{n}{2}+\frac{m}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{\frac{n}{2}-\frac{m}{2}} \sqrt{\frac{n}{2}-\frac{m}{2}}}$$

$$\sqrt{\frac{n}{2}+\frac{m}{2}}\sqrt{\frac{n}{2}-\frac{m}{2}} = \sqrt{\frac{n^2}{4}-\frac{m^2}{4}}$$

$$\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{n}{2}+\frac{m}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{\frac{n}{2}-\frac{m}{2}}$$

$$= \underbrace{\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{n}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{\frac{n}{2}}}_{\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{\frac{n}{2}}} \underbrace{\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{m}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{-\frac{m}{2}}}_{\left(\frac{\frac{n}{\cancel{2}}+\frac{m}{\cancel{2}}}{\frac{n}{\cancel{2}}-\frac{m}{\cancel{2}}}\right)^{m/2}}$$

$$\therefore \boxed{P(m,n) = \frac{n^n \sqrt{n}\, 2^{-n}}{\sqrt{2\pi}\sqrt{\frac{n^2}{4}-\frac{m^2}{4}}\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{n/2}\left(\frac{n+m}{n-m}\right)^{m/2}}}$$

continue w/ this next time...