PHYS 232 Feb. 7, 2025

- Assignment #2 due $\boxed{\text{now}}$
- No office hours today

Last Time: Binomial Dist'n

$$\boxed{P_B(x;n,p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}}$$

Today: Derive the Poisson distribution as a special limit of the Binomial dist'n.

avg. no. of successes

\# trials

look at limit $\mu \ll n$

only a few successes in a large no. of trials.

Since $\mu = np$

$$\mu \ll n \Rightarrow \cancel{n}p \ll \cancel{n}$$

$$\Rightarrow p \ll 1 \quad (\text{low prob. of success})$$

Eg. Counting Experiment 1.

Count no. of muon decays in time interval Δt.
Repeat many times. Usually get no decays for short Δt. $\Rightarrow$ Prob p of observing a muon decay is small.

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{1}{x!} \underbrace{\overbrace{\frac{n!}{(n-x)!} p^x}^{\mu^x}}_{①} \underbrace{\overbrace{(1-p)^{-x}}^{1}}_{②} \underbrace{\overbrace{(1-p)^n}^{e^{-\mu}}}_{③}$$

Start w/ ①.

$$\frac{n!}{(n-x)!} = \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{\cancel{(n-x)!}}$$

since $p \ll 1$, $x \ll n$ (few successes)

$$\therefore \left.\begin{aligned} n &= n \\ (n-1) &\approx n \\ (n-2) &\approx n \\ &\vdots \\ (n-x+1) &\approx n \end{aligned}\right\} x \text{ factors of } n \Rightarrow n^x$$

large no. of trials

small no. of successes

$$\therefore \frac{n!}{(n-x)!} \approx n^x \quad \therefore \frac{n!}{(n-x)!} p^x \approx (np)^x$$

$$\approx \boxed{\mu^x}$$

Next, consider 2

$$\boxed{(1-p)^{-x} \approx (1)^{-x} = 1} \text{ for } p \ll 1.$$

Finally 3 Since, in this case, $(1-p)$ is raised to a large power, we can't simply approx. $1-p \approx 1$. We need to be more careful.

$(1-p)^n$ know that $\mu = np \Rightarrow n = \frac{\mu}{p}$

$\downarrow$

$$(1-p)^{\mu/p} = \left[(1-p)^{\frac{1}{p}}\right]^{\mu}$$

Claim: $\lim_{p\to 0}\left[(1-p)^{\frac{1}{p}}\right] = \frac{1}{e}$ ← Euler's no.

$e \approx 2.718\ldots$

Proof: $y = (1-p)^{\frac{1}{p}}$

$$\ln y = \frac{1}{p}\underbrace{\ln(1-p)}_{\approx -p \text{ for } p \text{ small}}$$

$$\lim_{p\to 0} \ln y = \frac{1}{p}(-p) = -1$$

$\therefore y \approx e^{-1}$ when $p \ll 1$

$\therefore (1-p)^{\frac{1}{p}} \approx \frac{1}{e}$

$\therefore (1-p)^{n} = \left[(1-p)^{\frac{1}{p}}\right]^{\mu} \approx e^{-\mu}$ ∎

$\ln(1-x)$

$\approx -x$

for $|x| \ll 1$

∴ The Poisson distribution is:

$$\lim_{p \to 0} P_B(x; n, p) = \frac{1}{x!} \mu^x e^{-\mu}$$

$$\boxed{P_p(x; \mu) = \frac{\mu^x}{x!} e^{-\mu}}$$

Check the requirement that

$$\sum_{x=0}^{\infty} P_p = 1$$

here, the prob of a large no. of successes is very small & makes essentially no contribution to the sum.

$$\sum_{x=0}^{\infty} P_p(x; \mu) = \sum_{x=0}^{\infty} \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

$$= e^{-\mu} \underbrace{\left[1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \dots \right]}_{\text{Taylor series of } e^{\mu}}$$

$$\therefore \sum_{x=0}^{\infty} P_P(x;\mu) = e^{-\mu} e^{\mu} = 1 \quad \checkmark$$

Mean of Poisson Dist'n

$$\bar{x} = \sum_{x=0}^{\infty} x \, P_P(x;\mu)$$

$$= \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu} \left[0 + \sum_{x=1}^{\infty} \frac{x}{x!} \mu^x \right]$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \underbrace{\mu^x}_{\mu\mu^{x-1}}$$

$$= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \mu^{x-1}$$

sub in $y = x-1$ $\qquad x=1, \; y=0$

$\qquad x=\infty, \; y=\infty$

$$\bar{x} = \mu e^{-\mu} \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} \mu^y}_{e^{\mu} \text{ Taylor series.}}$$

$$\boxed{\therefore \bar{x} = \mu} \quad \text{as expected}$$

Standard Deviation

$$\sigma^2 = \overline{x^2} - \bar{x}^2$$

$$= \underbrace{\sum_{x=0}^{\infty} x^2 P_p(x;\mu)}_{\overline{x^2}} - \mu^2$$

In HW 3, you will show that:

$$\sigma^2 = \mu \Rightarrow \sigma = \sqrt{\mu}$$

Poisson dist'n is characterized by a single parameter μ:

$$\boxed{\begin{array}{l}\text{mean: } \mu \\ \text{std. dev. } \sigma = \sqrt{\mu}\end{array}}$$

Suppose we want to know prob. of getting between x_1 & x_2 counts during some counting experiment.

$$P_{x_1-x_2} = P_P(x_1;\mu) + P_P(x_1+1;\mu)$$

$$+ P_P(x_1+2;\mu) + \dots + P_P(x_2-1;\mu)$$

$$+ P_P(x_2;\mu)$$

$$= \sum_{x=x_1}^{x_2} P_P(x;\mu)$$

Example: Avg. no. of flaws in a fibre optic cable of length 50m is 1.2.

(i) What is the prob. of exactly 3 flaws in 150 m of cable?

$$P_p = \frac{\mu^x}{x!} e^{-\mu}$$

For 150 m of cable, expect avg. no. of flaws to be $\mu = 3(1.2) = 3.6$

If we require $x = 3$ flaws:

$$P_p = \frac{(3.6)^3}{3!} e^{-3.6} = \underline{\underline{0.212}}$$