PHYS 232 Jan. 31, 2025

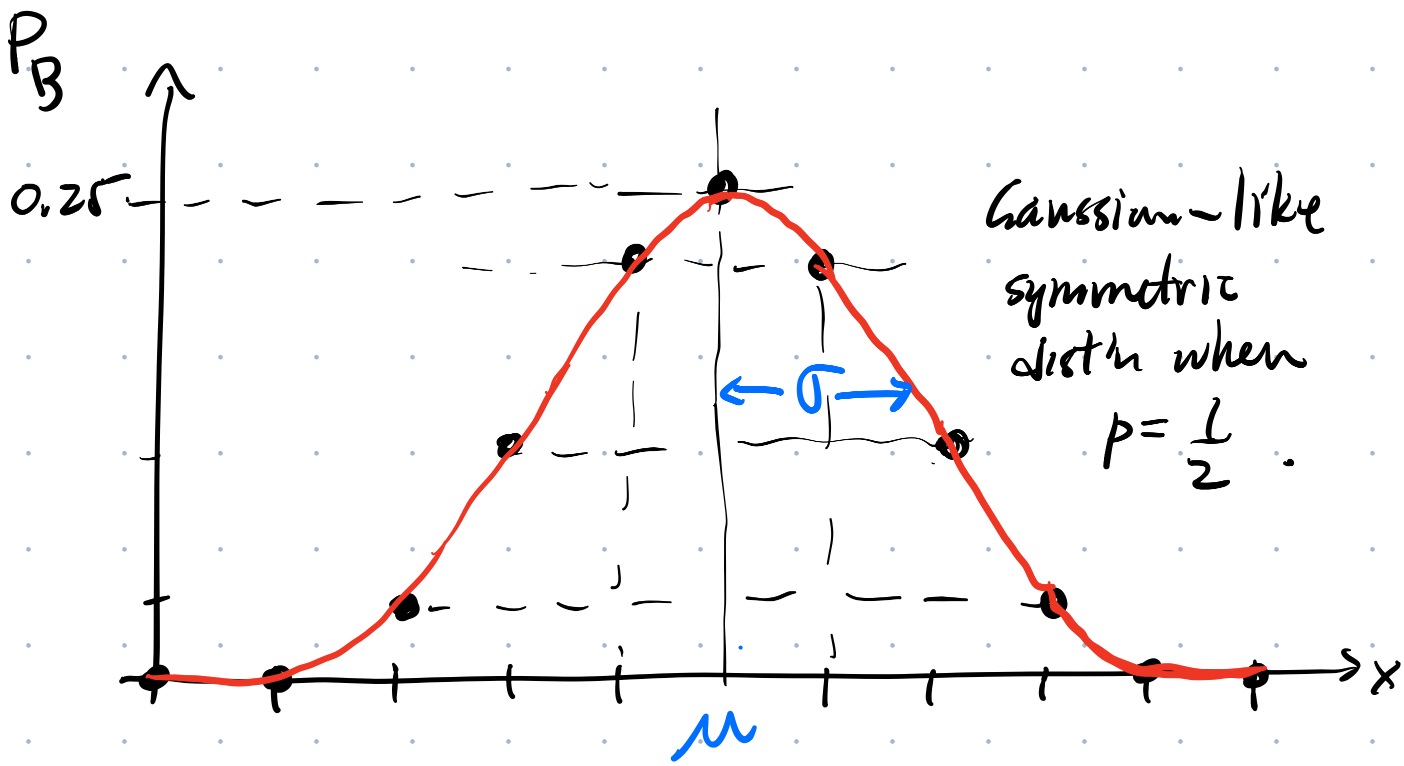
- Assignment #2 is on the course website.
  - due Feb. 7 @ 09:00

- Sign up for Experiment #3 by $\boxed{\text{today}}$ @ 23:59

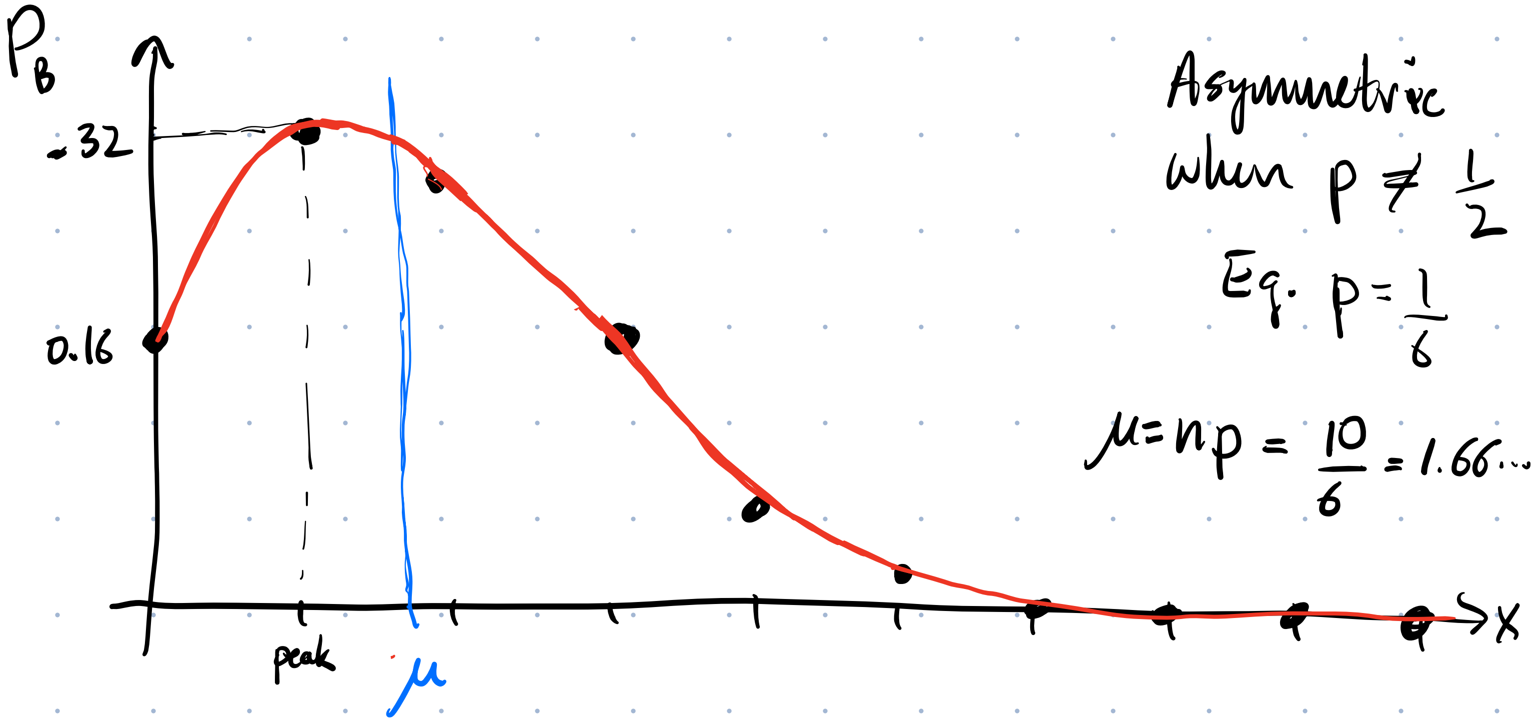
- No office hours Fri. Feb. 7. They will be on Thur. Feb 6 9:30–10:30 instead.

<u>Last Time:</u> Binomial Dist'n

$$\boxed{P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}}$$



Gaussian-like symmetric dist'n when $p = \frac{1}{2}$.



Asymmetric when $p \neq \frac{1}{2}$

Eq. $p = \frac{1}{6}$

$\mu = np = \frac{10}{6} = 1.66\ldots$

Today: Radom Walk.

p : prob. step right

$1-p$: prob. step left

If step size is ℓ, displacement is

$$\Delta x = \underbrace{x}_{\text{x steps right}} \ell - \underbrace{(n-x)}_{\text{n-x steps left}} \ell = 2x\ell - n\ell$$

$$= 2n\ell\left(\frac{x}{n} - \frac{1}{2}\right)$$

$$\therefore \overline{\Delta x} = 2nl\left(\frac{\mu}{n} - \frac{1}{2}\right) \qquad \text{but } \mu = np$$

$$\boxed{\therefore \overline{\Delta x} = 2nl\left(p - \frac{1}{2}\right)}$$

Next, calc. $\overline{\Delta x^2}$:

$$\overline{\Delta x^2} = \sum_{x=0}^{n} (\Delta x)^2 P_B(x; n, p)$$

First work out $\Delta x^2 = (2nl)^2\left(\frac{x}{n} - \frac{1}{2}\right)^2$

$$= 4n^2l^2\left(\frac{x^2}{n^2} - \frac{x}{n} + \frac{1}{4}\right)$$

$$= 4l^2\left(x^2 - nx + \frac{n^2}{4}\right)$$

$$\therefore \overline{\Delta x^2} = 4l^2 \sum_{x=0}^{n} \left(x^2 - nx + \frac{n^2}{4}\right) P_B(x; n, p)$$

$$\overline{\Delta x^2} = 4l^2 \left\{ \sum_{x=0}^{n} x^2 P_B - n \underbrace{\sum_{x=0}^{n} x P_B}_{\mu = np} + \frac{n^2}{4} \underbrace{\sum_{x=0}^{n} P_B}_{1} \right\}$$

$$\therefore \overline{\Delta x^2} = 4l^2 \left\{ \underbrace{\sum_{x=0}^{n} x^2 P_B}_{\overline{x^2}} - n^2 p + \frac{n^2}{4} \right\} \quad \#$$

To go further, we need to focus on evaluating the first sum.

$$\overline{x^2} = \sum_{x=0}^{n} x^2 P_B = \cancelto{0}{0^2 P_B} + \sum_{x=1}^{n} x^2 P_B$$

$$\overline{x^2} = \sum_{x=1}^{n} x^2 \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

- $$\frac{x^2}{x!} = \frac{x^{\cancel{2}}}{\cancel{x}(x-1)!} = \frac{x}{(x-1)!}$$

- $$n! p^x = n(n-1)! p p^{x-1}$$
$$= np(n-1)! p^{x-1}$$

$$\therefore\ \overline{x^2} = np \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1}(1-p)^{n-x}$$

sub $\quad y = x-1 \quad \therefore \quad x = y+1$

when $\quad x=1 \quad y=0$

$\qquad x=n \quad y=n-1$

$$\therefore\ \overline{x^2} = np \sum_{y=0}^{n-1} (y+1) \frac{(n-1)!}{y!(n-1-y)!} p^{y}(1-p)^{n-1-y}$$

another sub : $m = n-1$

$$\overline{x^2} = np \sum_{y=0}^{m} (y+1) \underbrace{\frac{m!}{y!(m-y)!} p^{y}(1-p)^{m-y}}_{P_B(y;m,p)}$$

$$= np \left\{ \underbrace{\sum_{y=0}^{m} y\, P_B(y;m,p)}_{\bar{y} = mp = (n-1)p} + \underbrace{\sum_{y=0}^{m} P_B(y;m,p)}_{1} \right\}$$

$$\therefore \overline{x^2} = np\left\{(n-1)p+1\right\}$$

$$\boxed{\therefore \overline{x^2} = n^2p^2 - np^2 + np} \quad \circledast$$

Sub $\circledast$ into $\#$ to find $\overline{\Delta x^2}$ (avg. of the square of displacement for our random walk.)

$$\boxed{\overline{\Delta x^2} = 4l^2\left[\left(n^2p^2 - np^2 + np\right) - n^2p + \frac{n^2}{4}\right]}$$

Complicate. Not much intuition can be gleaned for this expression.

Consider the special case of $p = \frac{1}{2}$

$$\overline{\Delta x^2} = 4l^2\left[\cancel{\frac{n^2}{4}} \underbrace{- \frac{n}{4} + \frac{n}{2}}_{\frac{n}{4}} \cancel{- \frac{n^2}{2}} + \cancel{\frac{n^2}{4}}\right]$$

$$\boxed{\overline{\Delta x^2} = n l^2}$$

$\overline{\Delta x} = 0$ for $p = \frac{1}{2}$

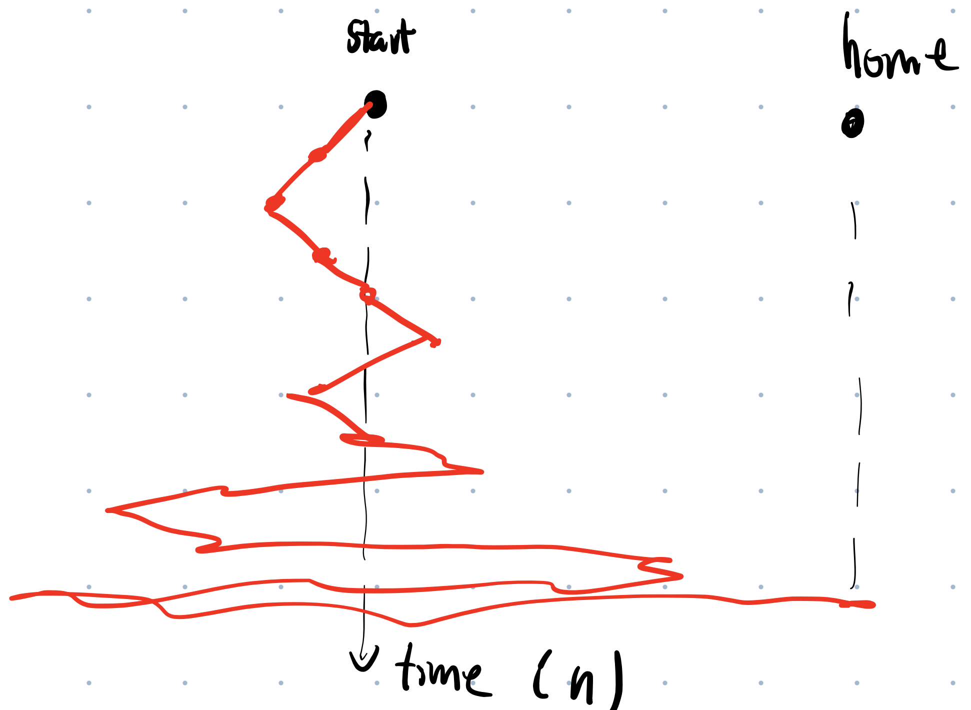
$\therefore$ Avg. displacement is zero when $p = \frac{1}{2}$, but avg. of the square of the displacement is prop. to n.

or $\sqrt{\overline{\Delta x^2}} = \sqrt{n}\, l$

Recall that the std. dev. could be expressed as

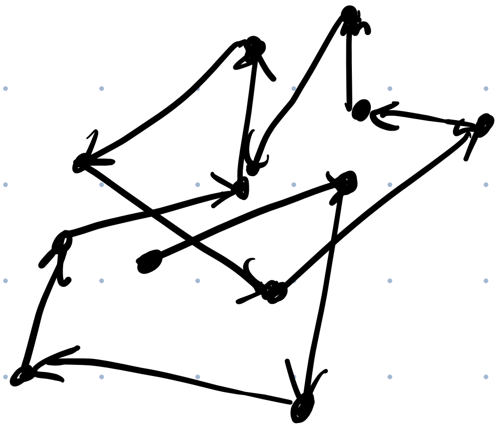
$$\sigma_{\Delta x} = \sqrt{\overline{\Delta x^2} - \overline{\Delta x}^2}$$

$$= \sqrt{n}\, l \text{ for } p = \frac{1}{2}.$$



Return to $\overline{\Delta x^2} = n l^2$

This result also applies to Brownian motion of suspended particles.



here l is the avg. distance traveled by a particle between scattering events. l is called the "mean-free path".

If average time between scattering events is τ, then total time for n scatters will be

$$t = n\tau \Rightarrow n = \frac{t}{\tau}$$ ← total time

$$\overline{\Delta x^2} = n l^2 = \frac{t\, l^2}{\tau}$$

It is common to define a diffusion coefficient:

$$D \equiv \frac{l^2}{2\tau} \Rightarrow \overline{\Delta x^2} = 2Dt$$

For a spherical particle suspended in a fluid, the diffusion coefficient can be calculate & is given by:

$$D = \frac{k_B T}{6\pi\eta a} = \frac{RT}{6\pi\eta N_A a}$$

T: temp.

η: viscosity of fluid

a: particle radius

$$\therefore \quad \overline{\Delta x^2} = 2Dt = \left(\frac{2RT}{6\pi\eta N_A a}\right) t$$