PHYS 232 Jan. 31, 2025

- Assignment #2 is on the course website.
  - due Feb. 7 @ 09:00

- Sign up for Experiment #3 by Wednesday, Feb. 5 @ 23:59

<u>Last Time:</u> Binomial Dist'n

$$\boxed{P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}}$$

Prob. of obtaining x "successes" out of n trials when the prob. of success in any individual trial is p.

<u>Today:</u> Show that the mean & std. dev. of the binomial dist'n are given by:

$$\mu = np \qquad \sigma^2 = np(1-p)$$

Recall : $\mu = \sum_j x_j P(x_j)$

$$\sigma^2 = \sum_j x_j^2 P(x_j) - \mu^2$$

sum over
all possible
outcomes

Possible outcomes from n trials:

- $x=0$ no successes
- $x=1$ one success
- $x=2$ two successes

$\vdots$

- $x=n$ n successes

$$\mu = \sum_{x=0}^{n} x P_B(x)$$

$$\mu = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

expect to find
$\mu = np$

- write $\sum_{x=0}^{n} x(\text{stuff}) = \cancel{0(\text{stuff})} + \sum_{x=1}^{n} x(\text{stuff})$

- $x! = x(x-1)!$

$$\mu = \sum_{x=1}^{n} \cancel{x} \frac{n!}{\cancel{x}(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

- $p^x = p(p^{x-1})$

$n! = n(n-1)!$

$$\mu = \sum_{x=1}^{n} \boxed{np} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

$$\mu = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

make a substitution (like we do for integration)

$$y = x - 1 \Rightarrow x = y + 1$$

summation limits:

when $x = 1$, $y = 0$

$x = n$, $y = n-1$

$$\mu = np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!\,[n-(y+1)]!} p^y (1-p)^{n-(y+1)}$$

$$= np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!\,[(n-1)-y]!} p^y (1-p)^{(n-1)-y}$$

another subsititution: $m = n-1$

$$\therefore \mu = np \sum_{y=0}^{m} \underbrace{\frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y}}$$

$P_B(y; m, p)$ Binomial dist'n in terms of variables y, m.

$$\mu = np \underbrace{\sum_{y=0}^{m} P_B(y; m, p)}_{1 \text{ sum over all possible outcomes}}$$

$$\boxed{\therefore \mu = np}$$ as expected!

Exercise for the student:

Show that

$$\sigma^2 = np(1-p)$$

Know that $$\sigma^2 = \underbrace{\sum_j x_j^2 P(x_j)}_{\overline{x^2}} - \mu^2$$

Eg. Binomial dist'n application – Random walk.

Drunk starts @ stop sign. Has prob. p of taking a step right & prob. $q = 1-p$ of stepping left. After n steps of size Δx, what is the avg. dist. from sign? What is the std. dev.?

Avg. no. of steps right $\mu_R = np$
Avg. displacement right : $np\,\Delta x$

Avg. no. steps left : $\mu_L = n(1-p)$
Avg. displacement left : $-n(1-p)\Delta x$

∴ net displacement : $np\Delta x - n(1-p)\Delta x$

$$2np\Delta x - n\Delta x$$

$$\boxed{= 2n\Delta x\left(p - \frac{1}{2}\right)}$$

If $p = \frac{1}{2}$, net displacement is zero as expected.

$\sigma^2 = np(1-p)$ spread in no. of steps right grows as $\sqrt{n}$

Let's plot some P_B dist'ns

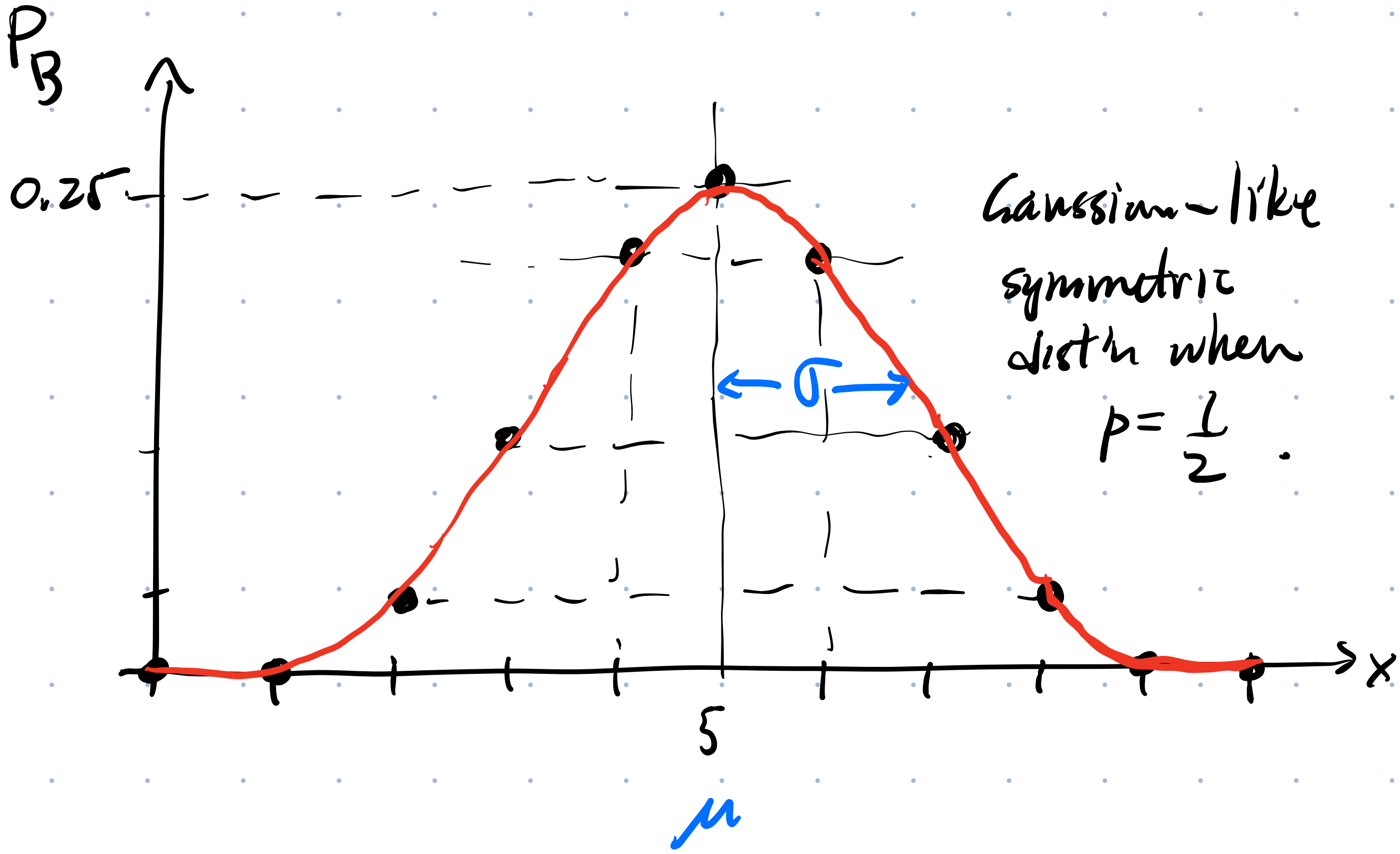
$$P_B = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

If $p = q = \frac{1}{2}$ $n = 10$

x	$P_B(x; 10, \frac{1}{2})$
0	9.8×10^{-4}
1	9.8×10^{-3}
2	4.4×10^{-2}
3	0.12
4	0.21
5	0.25
6	0.21
7	0.12
8	4.4×10^{-2}
9	9.8×10^{-3}
10	9.8×10^{-4}

$$= \left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}$$

symmetry when $p = \frac{1}{2}$.



Gaussian-like symmetric dist'n when $p = \frac{1}{2}$.

If try $p = \frac{1}{2}$ $\quad q = 1 - p = \frac{5}{6}$

x	$P_B(x; 10, \frac{1}{6})$
0	$\left(\frac{5}{6}\right)^{10} = 0.16$
1	0.32 ← peak
2	0.29
3	0.16
4	0.054
⋮	
10	$\left(\frac{1}{6}\right)^{10} = 1.6 \times 10^{-8}$

$$\mu = np = 10\left(\frac{1}{6}\right) = 1\tfrac{4}{6} = 1\tfrac{2}{3} = 1.66\cdots$$

This time, we get an asymmetric dist'n with μ offset from the peak.