PHYS 232 Jan. 24, 2025

Last Time:

Discrete distributions:

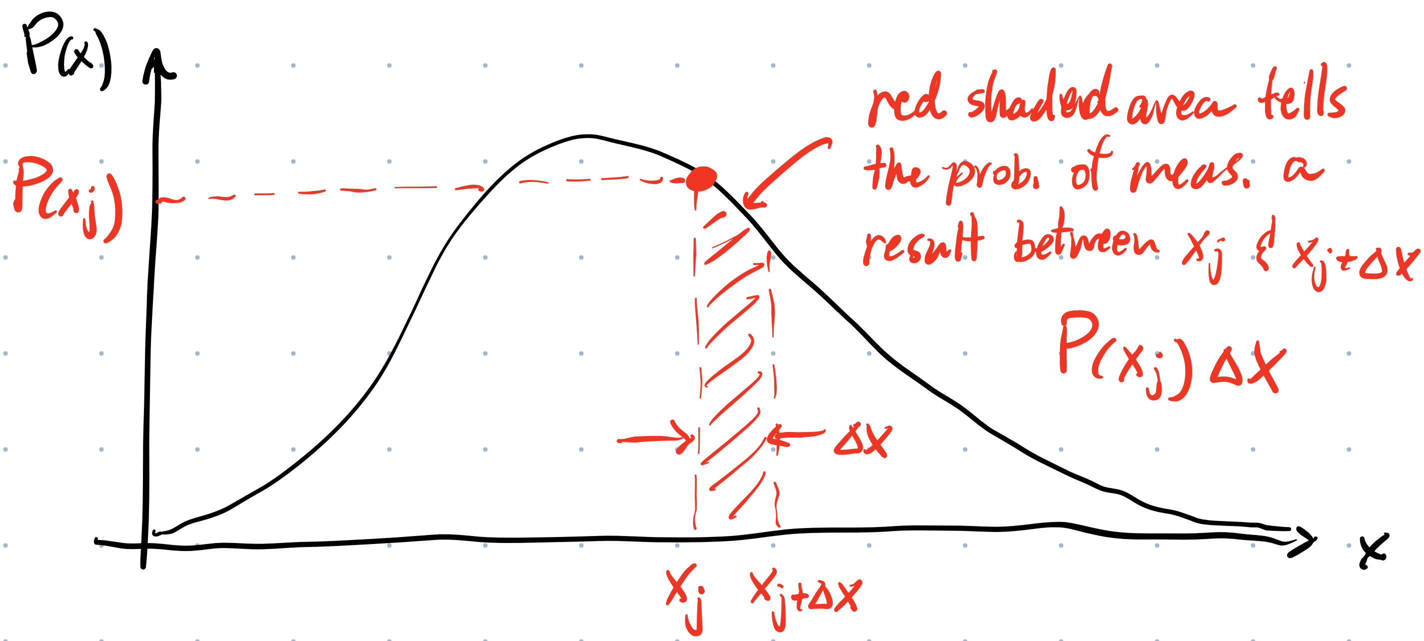
Result of a measurement must be one of n possible outcomes. Prob. of getting outcome x_j is $P(x_j)$ where $j = 1 \dots n$.

$$\mu = \sum_{j=1}^{n} x_j P(x_j) \qquad P(x_j) \text{ is unitless}$$

$$\sigma^2 = \sum_{j=1}^{n} (x_j - \mu)^2 P(x_j)$$

$$= \sum_{j=1}^{n} x_j^2 P(x_j) - \mu^2 = \overline{x^2} - \overline{x}^2$$

Continuous Distributions



$P(x)$ is a prob. density. Units of $P(x)$ are $\frac{1}{[x]}$.

For a normalized prob. density, $\int_{-\infty}^{\infty} P(x)\,dx = 1$.

$$\mu = \int_{-\infty}^{\infty} x\,P(x)\,dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 P(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 P(x)\,dx - \mu^2 = \overline{x^2} - \overline{x}^2$$

Example: Discrete Distribution

Invent a dice game.

Roll a pair of dice $\quad D_1 = 1, 2, 3, 4, 5, 6$

$D_2 = 1, 2, 3, 4, 5, 6$

outcome ① $\quad$ If $|D_1 - D_2| \geq 3$

then you win $M_1 = \$1$.

outcome ② $\quad$ If $1 \leq |D_1 - D_2| \leq 2$

then you win $M_2 = \$2$

outcome ③ $\quad$ If $D_1 = D_2$ or $D_1 - D_2 = 0$

then you win $M_3 = \$3$

It costs \$1.75 for each play

If $\overline{M} > \$1.75$ should play as much as possible

Calculate $\bar{m} = \sum_{j=1}^{3} m_j P(m_j)$

Need to find $P(m_j)$ for each possible case.

Start w/ $D_1 = D_2$ case 3

$$P_{3} = P(M_3) = (1 \text{ and } 1) \text{ or } (2 \text{ and } 2) \text{ or } (3 \text{ and } 3) \cdots \text{ or } (6 \text{ and } 6)$$

$$= \left(\frac{1}{6} \cdot \frac{1}{6}\right) + \left(\frac{1}{6} \cdot \frac{1}{6}\right) + \left(\frac{1}{6} \cdot \frac{1}{6}\right) \cdots + \left(\frac{1}{6} \cdot \frac{1}{6}\right)$$

$$= 6\left(\frac{1}{36}\right) = \frac{1}{6}$$

$$\boxed{P_{3} = \frac{1}{6}}$$

Try $P_{②} = P(M_2)$

$= (1 \text{ and } 2) \text{ or } (1 \text{ and } 3)$

$\text{or } (2 \text{ and } 1) \text{ or } (2 \text{ and } 3) \text{ or } (2 \text{ and } 4)$

$\text{or } (3 \text{ and } 1) \text{ or } (3 \text{ and } 2) \text{ or } (3 \text{ and } 4) \text{ or } (3 \text{ and } 5)$

$\text{or } (4 \text{ and } 2) \text{ or } (4 \text{ and } 3) \text{ or } (4 \text{ and } 5) \text{ or } (4 \text{ and } 6)$

$\text{or } (5 \text{ and } 3) \text{ or } (5 \text{ and } 4) \text{ or } (5 \text{ and } 6)$

$\text{or } (6 \text{ and } 4) \text{ or } (6 \text{ and } 5)$

$$= 2\left[\frac{1}{36} + \frac{1}{36}\right] + 2\left[\frac{3}{36}\right] + 2\left[\frac{4}{36}\right]$$

$$= \frac{18}{36} \Rightarrow \boxed{P_{②} = \frac{1}{2}}$$

$$P_{①} = 1 - P_{②} - P_{③} = 1 - \frac{1}{2} - \frac{1}{6}$$

$$= \frac{6-3-1}{6} = \frac{1}{3} \qquad \boxed{P_{①} = \frac{1}{3}}$$

Another visual way to get these same prob. is by charting all possible outcomes of a roll.

D_1 / D_2	1	2	3	4	5	6
6	5	4	3	2	1	0
5	4	3	2	1	0	1
4	3	2	1	0	1	2
3	2	1	0	1	2	3
2	1	0	1	2	3	4
1	0	1	2	3	4	5

$D_1 - D_2 = 0$, have 6 of 36 cases $\therefore P_3 = \frac{1}{6}$

$1 \leq |D_1 - D_2| \leq 2$, have 18 or 36 cases $\therefore P_2 = \frac{1}{2}$

$|D_1 - D_2| \geq 3$, have 12 or 36 cases $\therefore P_1 = \frac{1}{3}$

$$\bar{M} = \sum_{j=1}^{3} M_j P(M_j) = 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{6}\right)$$

$$\bar{M} = \frac{2+6+3}{6} = \frac{11}{6} = 1\tfrac{5}{6}$$

$$\bar{M} = \$1.83333\ldots > \$1.75$$

profitable!

<u>Standard Deviation</u>

$$\sigma^2 = \sum_{j=1}^{3} (M_j - \bar{M})^2 P(M_j)$$

$$= \left(1 - 1\tfrac{5}{6}\right)^2 \frac{1}{3}$$

$$+ \left(2 - 1\tfrac{5}{6}\right)^2 \frac{1}{2}$$

$$+ \left(3 - 1\tfrac{5}{6}\right)^2 \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^2 \frac{1}{3} + \left(\frac{1}{6}\right)^2 \frac{1}{2} + \left(\frac{7}{6}\right)^2 \frac{1}{6}$$

$$= \frac{25}{36}\,\frac{1}{3} + \frac{1}{36}\,\frac{1}{2} + \frac{49}{36}\,\frac{1}{6}$$

$$= \frac{25}{36}\,\frac{2}{6} + \frac{1}{36}\,\frac{3}{6} + \frac{49}{36}\,\frac{1}{6}$$

$$= \frac{50+3+49}{6\cdot 36} = \frac{102}{6\cdot 36} = \frac{17}{36} = \sigma^2$$

$$\text{or} \quad \sigma = \frac{\sqrt{17}}{6} = 0.687.$$

Continuous Dist'n Example

Prob. that an electron is a dist. r from the nucleus of a hydrogen atom (proton) is:

$$P(r)\,dr = C r^2 e^{-r/R}\,dr$$

where C and R are constants.

(a) Use the requirement $\int_{r=0}^{\infty} P(r)dr = 1$

to find the value of C.

$$\int_{r=0}^{\infty} Cr^2 e^{-r/R} dr = 1$$

$$\therefore \frac{1}{C} = \int_0^{\infty} r^2 e^{-r/R} dr$$

use Integration by parts (IBP)

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$u = r^2 \quad du = 2r\,dr$

$dv = e^{-r/R} dr \quad v = -Re^{-r/R}$

0

$$\frac{1}{C} = -r^2 Re^{-r/R}\Big|_0^{\infty} + 2R\int_a^b re^{-r/R} dr$$

IBP again.

$u = r \quad du = dr$

$dv = e^{-r/R} dr \quad v = -Re^{-r/R}$

$$\frac{1}{C} = 2R\left[\xrightarrow{0}\cancel{-rRe^{-rR}\Big|_0^{\infty}} + R\int_0^{\infty} e^{-r/R}\,dr \right]$$

$$= 2R^2 (-R)e^{-r/R}\Big|_0^{\infty}$$

$$= -2R^3\left[\cancel{e^{-\infty}}^{\to 0} - \underbrace{e^{0}}_{1}\right] = 2R^3$$

$$\boxed{\therefore\ C = \frac{1}{2R^3}}$$

Probability density normalization constant. This value of C gaurantees that the area under $P(r)$ is 1.

(b) Find the average dist $\bar{r}$ that the e^- is from the proton.

$$\bar{r} = \int_0^{\infty} r\,P(r)\,dr = \int_0^{\infty} r\Big[\underbrace{\frac{1}{2R^3} r^2 e^{-r/R}}_{P(r)}\Big]dr$$

$$\therefore \bar{r} = \frac{1}{2R^3}\int_0^\infty r^3 e^{-r/R}\,dr$$ IBP again...

$$u = r^3 \quad du = 3r^2 dr$$
$$dv = e^{-r/R}dr \quad v = -Re^{-r/R}$$

$$\bar{r} = \frac{1}{2R^3}\left[\cancel{-r^3Re^{-r/R}}^{\to 0}\Big|_0^\infty + 3R\int_0^\infty r^2 e^{-r/R}dr\right]$$

$$= \frac{3}{2R^2}\underbrace{\int_0^\infty r^2 e^{-r/R}dr}_{= \frac{1}{C} = 2R^3}$$

$= \frac{1}{C} = 2R^3$ from first part of the problem.

$$\boxed{\bar{r} = \frac{3}{2R^2}\,2R^3 = 3R}$$

Find σ for electron in H atom

$$\sigma^2 = \int_0^\infty (r-\bar{r})^2 P(r)\,dr$$

$$= \int_0^\infty (r^2 - 2\bar{r}r + \bar{r}^2) P(r)\,dr$$

$$= \int_0^\infty r^2 P(r)\,dr - 2\bar{r}\underbrace{\int_0^\infty r P(r)\,dr}_{\bar{r}}$$

$$+ \bar{r}^2 \underbrace{\int_0^\infty P(r)\,dr}_{1}$$

$$= \underbrace{\int_0^\infty r^2 C r^2 e^{-r/R}\,dr}_{\text{solve by I.B.P.}} - 2\bar{r}^2 + \bar{r}^2$$

Exercise for the student: Show that $\sigma = \sqrt{3}\,R$