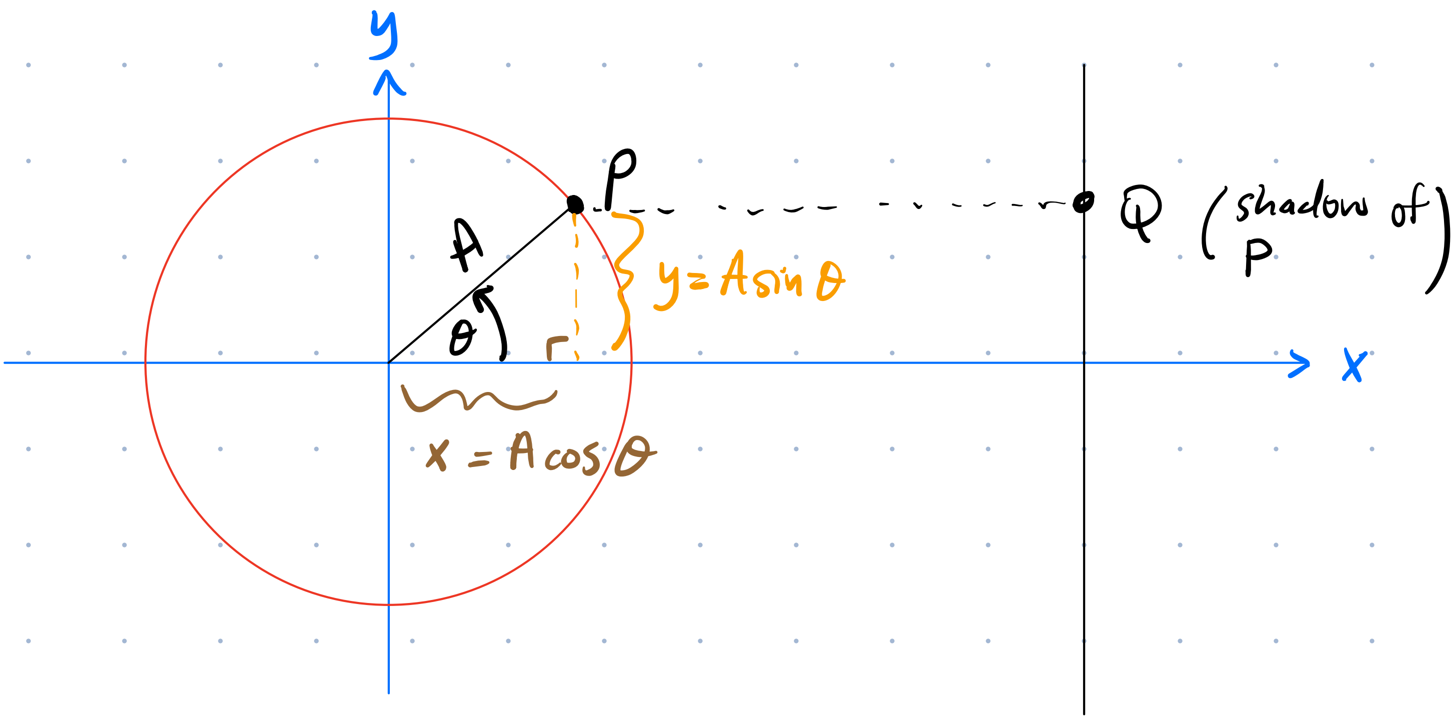
PHYS 232 — Jan. 10, 2025

<u>To do:</u>

- Sign up for Experiment #1 by Friday, January 10 @ 10:30 am. Link to sign up sheet is in Canvas.

- Complete Pre-lab before start of Experiment #1. Submit on loose paper (not in lab notebook).

Start w/ a review of <u>Simple Harmonic Motion</u>



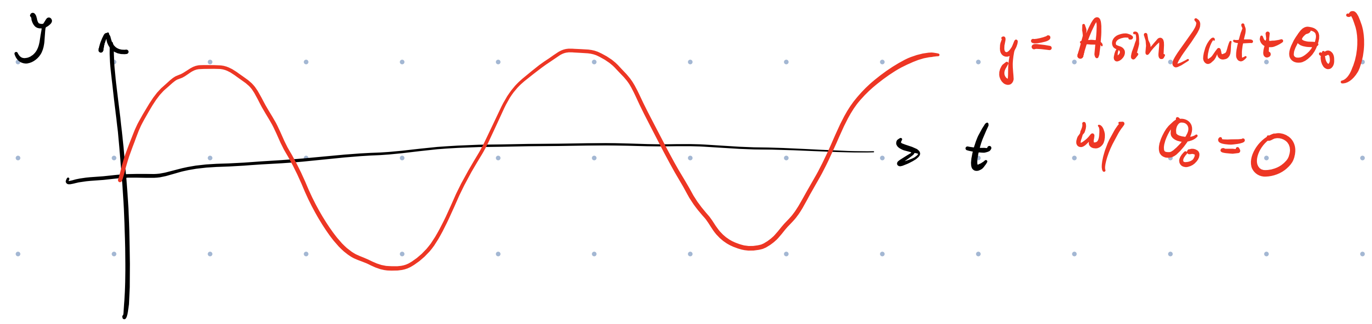
take $\theta = \omega t + \theta_0$ for uniform circular motion

$\uparrow$ angular freq. $\omega = 2\pi f = \frac{2\pi}{T}$

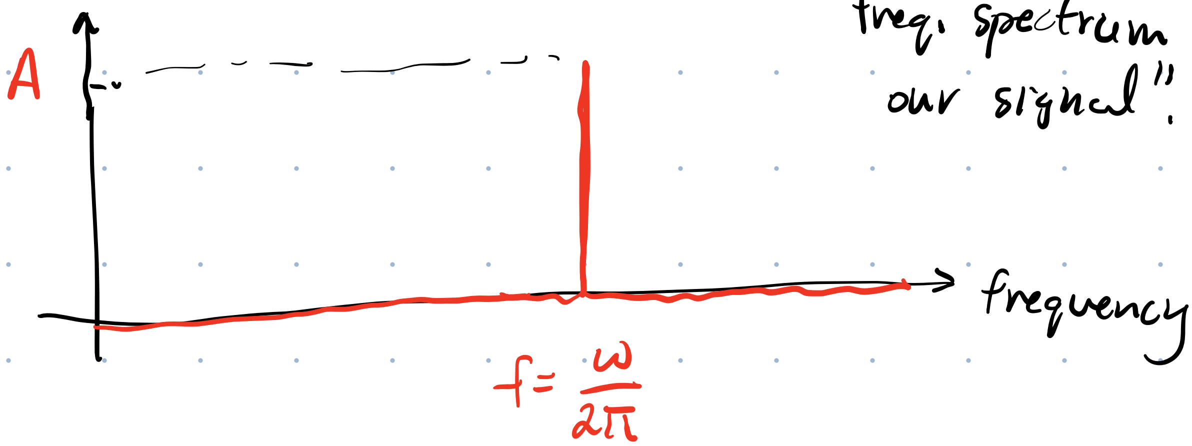
$$x = A\cos\theta = A\cos(\omega t + \theta_0)$$
$$y = A\sin\theta = A\sin(\omega t + \theta_0)$$
Gives the motion of Q which is the shadow of P.

Simple Harmonic motion describes the back and forth (sinusoidal) motion of a pt.

Time response of x or y



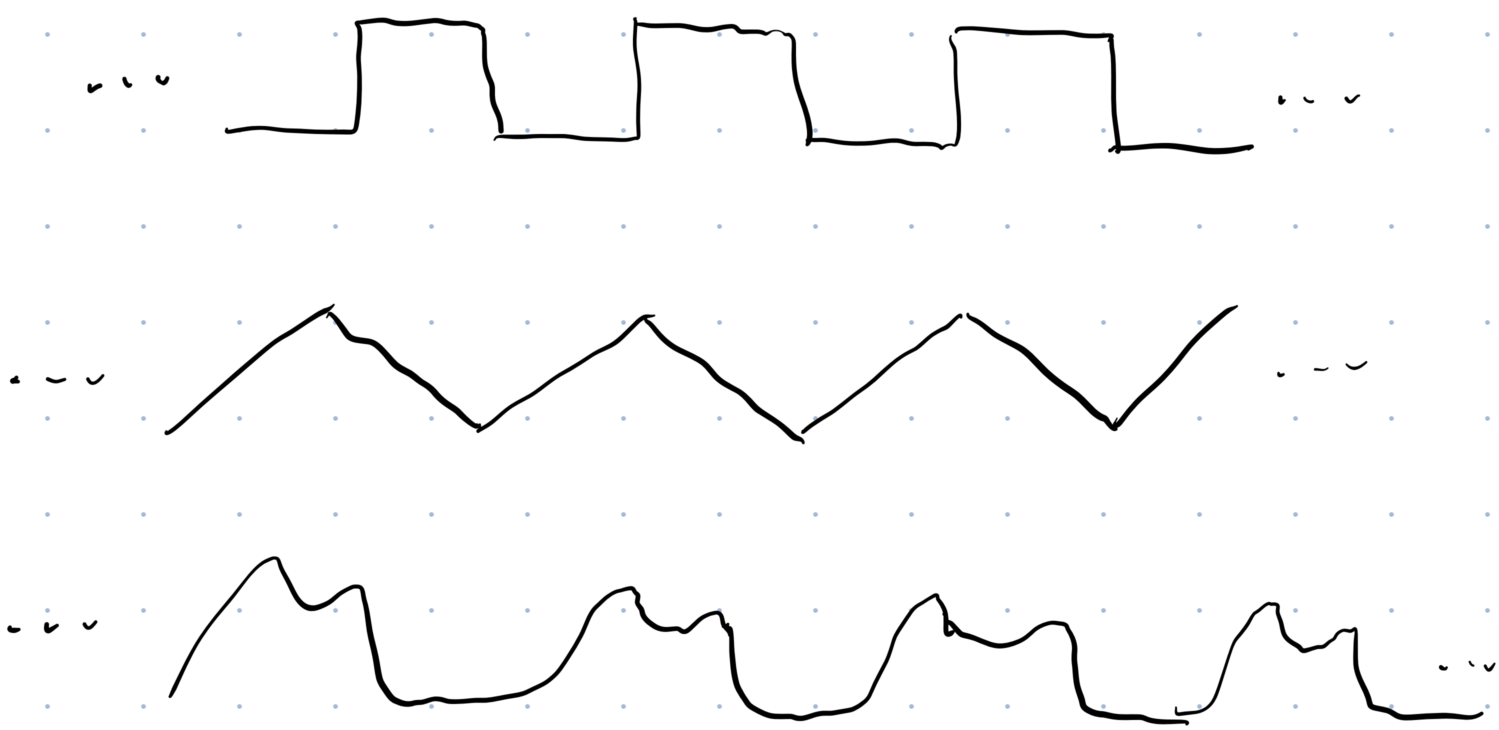
We could also ask what frequencies are contained in a signal.



"freq. spectrum of our signal".

For our pure sine wave, only one freq contained in the signal.

Goal: Express a periodic fcn:



as a series (sum) of pure sine & cosine fcns.

We will attempt to express a periodic fcn $f(x)$ in the form:

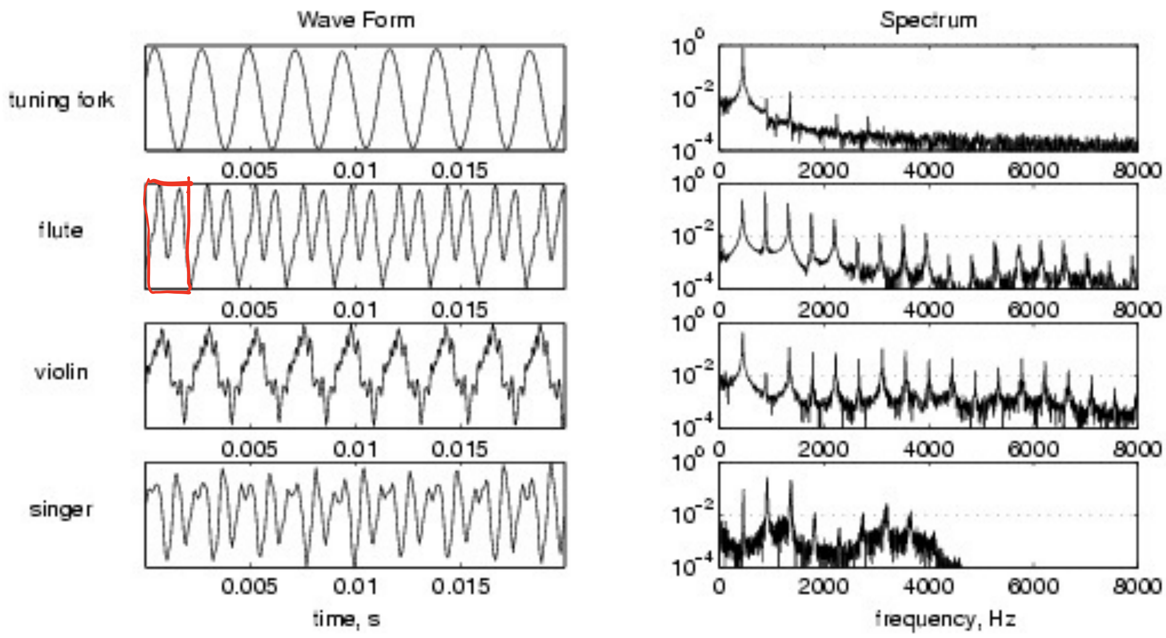
$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Need a porscription for finding the a_n & b_n coefficients.

Waveforms of various instruments.

Taken from:
https://amath.colorado.edu/pub/matlab/music



waveforms (pressure vs time signals) from various instruments playing the same note.

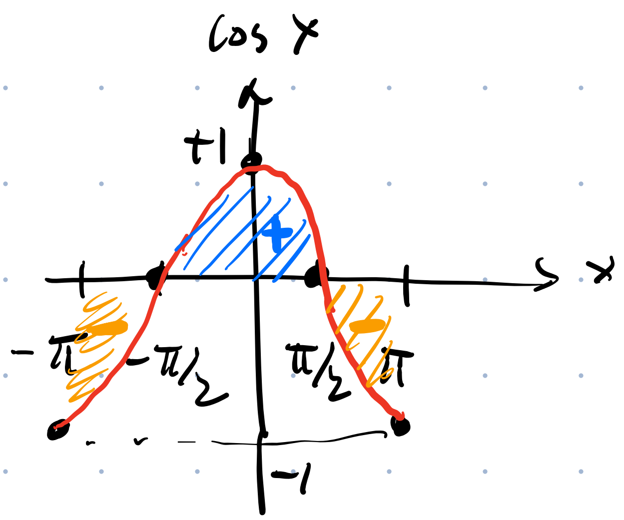
"frequency spectrum" of the instrument recordings. Shows at which frequencies the air pressure is oscillating and the relative strengths of the various frequency components.

Fourier Series: $f(x) = \boxed{\begin{aligned} &\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots \\ &+ b_1 \sin x + b_2 \sin 2x + \dots \end{aligned}}$

To find a_0, let's try evaluating the following integral:

$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ when the interval $-\pi$ to π corresponds to one period of $f(x)$.

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx}_{\pi a_0} + \cancelto{0}{\int_{-\pi}^{\pi} a_1 \cos x\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} a_2 \cos 2x\,dx} + \dots + \cancelto{0}{\int_{-\pi}^{\pi} b_1 \sin x\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} b_2 \sin 2x\,dx} + \dots \right]$$



$$\boxed{\therefore \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0}$$

Claim:
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

n: integer

$1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

When we sub in the Fourier series for $f(x)$ into the a_n & b_n integrals, we end w/ things like:

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx$$

As an example, consider

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$

Easiest to evaluate if we express the trig functions as complex exponentials.

Euler's Eq'n: $e^{\pm jx} = \cos x \pm j \sin x$

$$\therefore e^{jx} + e^{-jx} = 2\cos x$$

$$\boxed{\therefore \cos x = \frac{e^{jx} + e^{-jx}}{2}}$$

Similarly

$$e^{jx} - e^{-jx} = +2j\sin x$$

$$\boxed{\therefore \sin x = \frac{e^{jx} - e^{-jx}}{2j}}$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{jmx} - e^{-jmx}}{2j}\right)\left(\frac{e^{jnx} + e^{-jnx}}{2}\right) dx$$

If we multiply out all the terms, we get 4 terms of the form

e^{jkx} where k is an integer.

$$k = m+n, \quad m-n, \quad n-m, \quad -n-m$$

Try $m \neq n$ case

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jkx} \, dx = \frac{1}{2\pi}\frac{1}{jk} e^{jkx}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\frac{1}{jk}\Big[\underbrace{e^{jk\pi} - e^{-jk\pi}}_{2j \sin k\pi = 0 \;\; \forall \text{ integer } k.}\Big]$$

$$\boxed{\therefore \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \text{when } m \neq n}$$

$m = n = 0$ case. $\sin mx = \sin 0 = 0$

$\cos nx = \cos 0 = 1$

$$\int_{-\pi}^{\pi} (0)(1) \, dx = 0$$

$$\boxed{\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \text{when } m = n = 0}$$

$m = n \neq 0$ case.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\underbrace{\frac{e^{jnx} - e^{-jnx}}{2j}}_{\sin mx \text{ w/ } m=n} \Bigg) \left(\frac{e^{jnx} + e^{-jnx}}{2} \right) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{2jnx} \cancel{+1} \cancel{-1} - e^{-2jnx}}{4j} \right) dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin 2nx}{2} \, dx = 0$$ since $\sin 2nx$ is anti-symmetric on $-\pi$ to π

$$\boxed{\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \text{ when } n = m \neq 0}$$

$$\boxed{① \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \forall \, m, n}$$

Can likewise show that:

② $$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \neq 0 \\ \frac{1}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

③ $$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \neq 0 \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$$