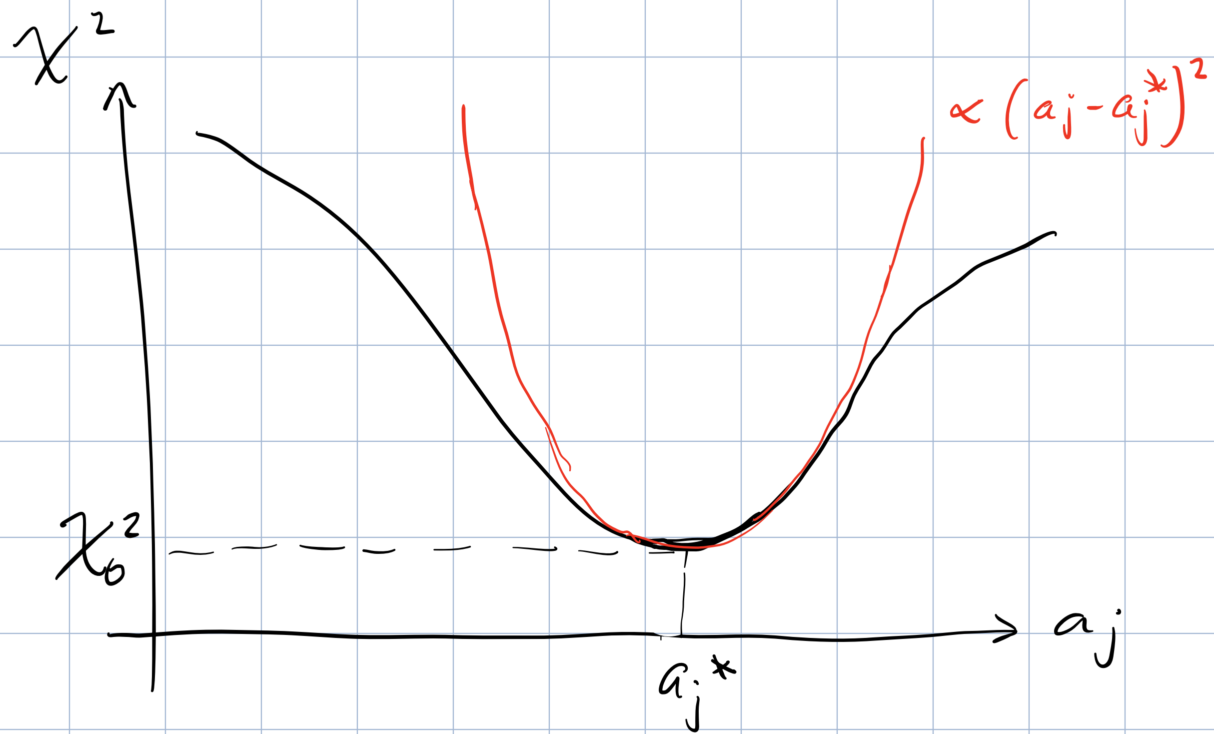
PHYS 232 April 3, 2024

- Assignment #6 due April 5 @ 11am
- Formal Report due Apr. 10 @ 11am

Last Time:

Near its minimum, χ^2 varies quadratically w.r.t. changes in the a_j parameter values



$$\chi^2 \simeq \chi_0^2 + B(a_j - a_j^*)^2$$

By finding χ^2 at 3 pts. near the minimum, can find:

$$\boxed{B = \frac{\chi_1^2 - 2\chi_2^2 + \chi_3^2}{2(\Delta a)^2}}$$

The uncertainty in parameter a_j is given by

$$\sigma_j = \frac{1}{\sqrt{B_j}}$$

$\therefore$ best-fit value for a_j is:

$$\boxed{a_j = a_j^* \pm \frac{1}{\sqrt{B_j}}}$$

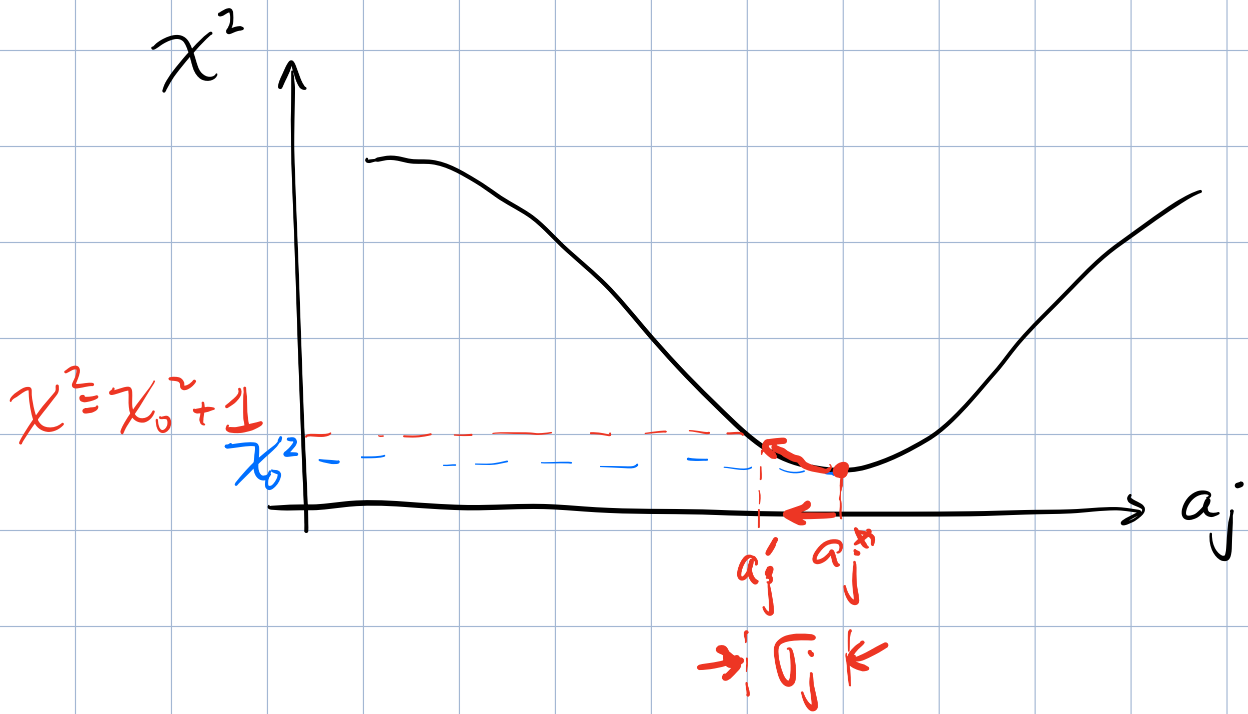
<u>Today</u> Develop some intuition about meaning of χ^2.

if $\sigma_j = \frac{1}{\sqrt{B}} \Rightarrow B = \frac{1}{\sigma_j^2}$

Notice if:

$$\chi^2 = \frac{(a_j - a_j^*)^2}{\sigma_j^2} + \chi_0^2$$ near the minimum of χ^2

Then, know that χ^2 will increase as we move a_j away from a_j^*.



Imagine moving along a_j axis starting at a_j^* & ending once χ^2 has increased from χ_0^2 to $\chi_0^2 + 1$.

$$\chi^2 = \frac{(a_j - a_j^*)^2}{\sigma_j^2} + \chi_0^2$$

$$\Downarrow$$

take $a_j \rightarrow a_j'$ s.t. $\chi^2 = \chi_0^2 + 1$

$$\underbrace{\cancel{\chi_0^2} + 1}_{\chi^2} = \frac{(a_j' - a_j^*)^2}{\sigma_j^2} + \cancel{\chi_0^2}$$

$$1 = \frac{(a_j' - a_j^*)^2}{\sigma_j^2}$$

$$\therefore \quad \sigma_j = |a_j' - a_j^*|$$

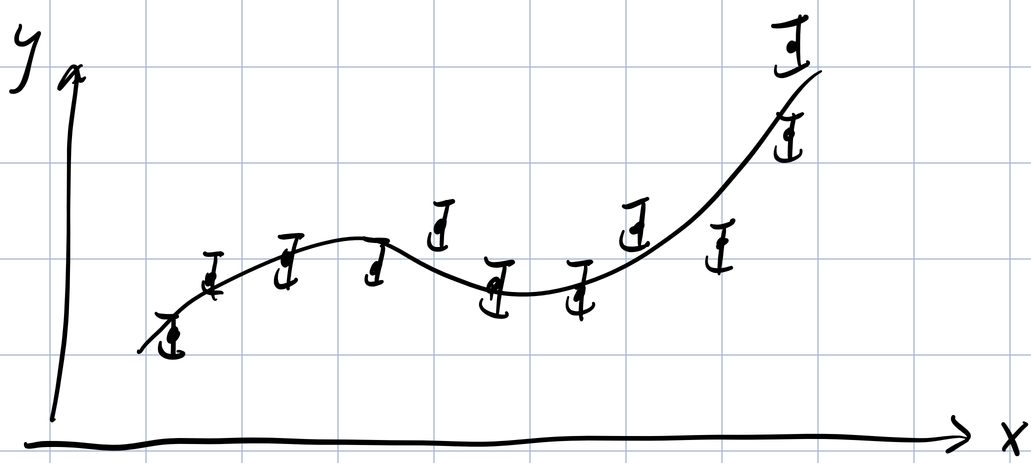
$$\boxed{\text{If we move away from the minimum of } \chi^2 \text{ along the } a_j \text{ axis s.t. } \chi^2 \to \chi_0^2 + 1, \text{ then the distance moved } |a_j' - a_j^*| \text{ is an estimate of the uncertainty in } a_j^*.}$$

What should we expect for a value of χ^2 after going a "good" fit?

Definition of χ^2:

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

assume we've measured $(x_i, y_i \pm \sigma_i)$ for $i = 1..N$.

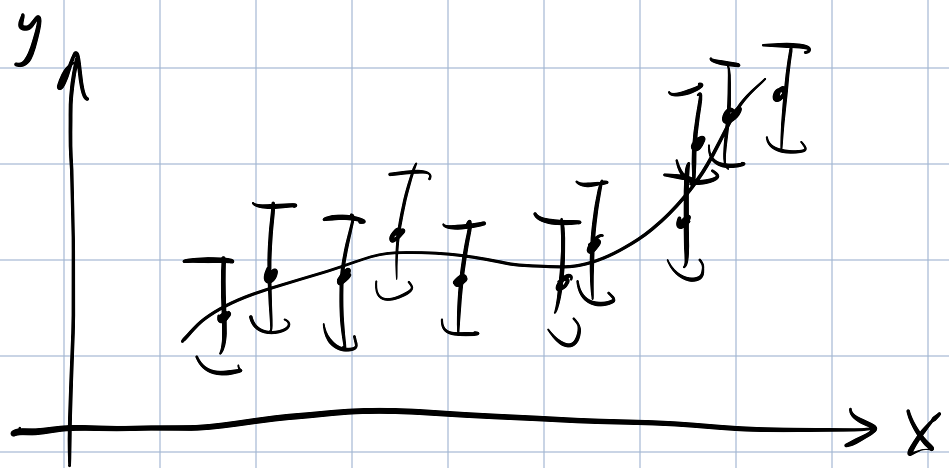


On average, we expect the deviation $y_i - y(x_i)$ to approximately equal to $\pm\sigma_i$ if we've made reasonable estimates of the uncertainties.

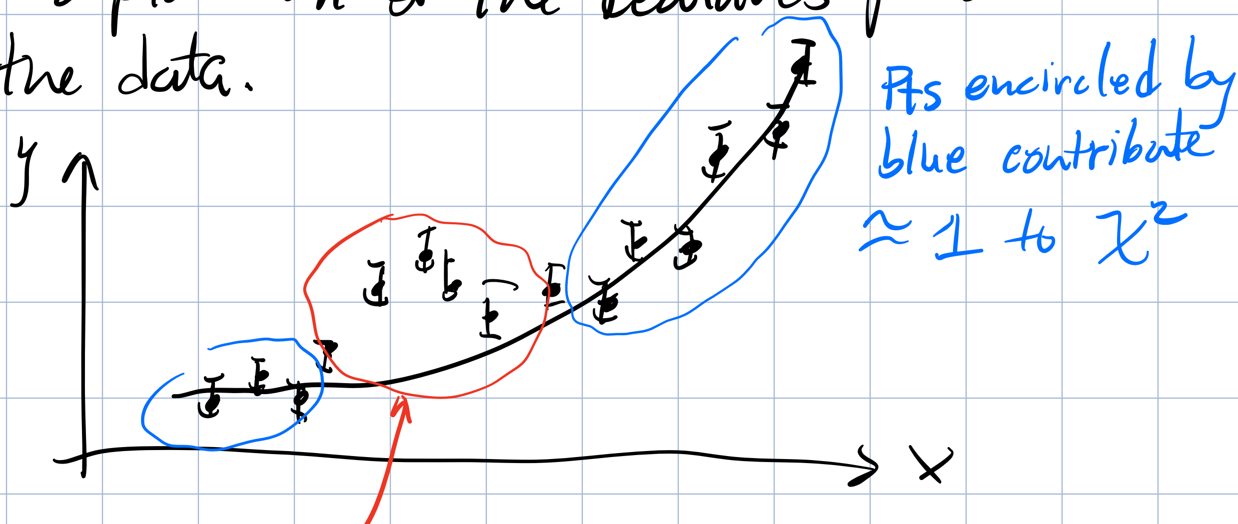
$$\therefore \chi^2 \approx \sum_{i=1}^{N} \left(\frac{\pm \cancel{\sigma_i}^{\,1}}{\cancel{\sigma_i}} \right)^2 \approx N$$

- For N large, a good fit to a data set will yield $\chi^2 \simeq N$.

- If you find that χ^2 is significantly less than N, it prob. means that σ_i has been over estimated.



- If χ^2 is larger than N, it could be an indication that your model does not capture all of the features present in the data.



Pts encircled by red, contribute more than 1 to χ^2

Overall $\chi^2 > N \longrightarrow$ indicates a poor fit (assumes that reasonable estimate of σ_i values)

$\rightarrow$ χ^2 is a "goodness of fit" parameter when σ_i estimated appropriately.

Expecting $\chi^2 \simeq N$ is correct when N is large. However, consider the following extreme cases.

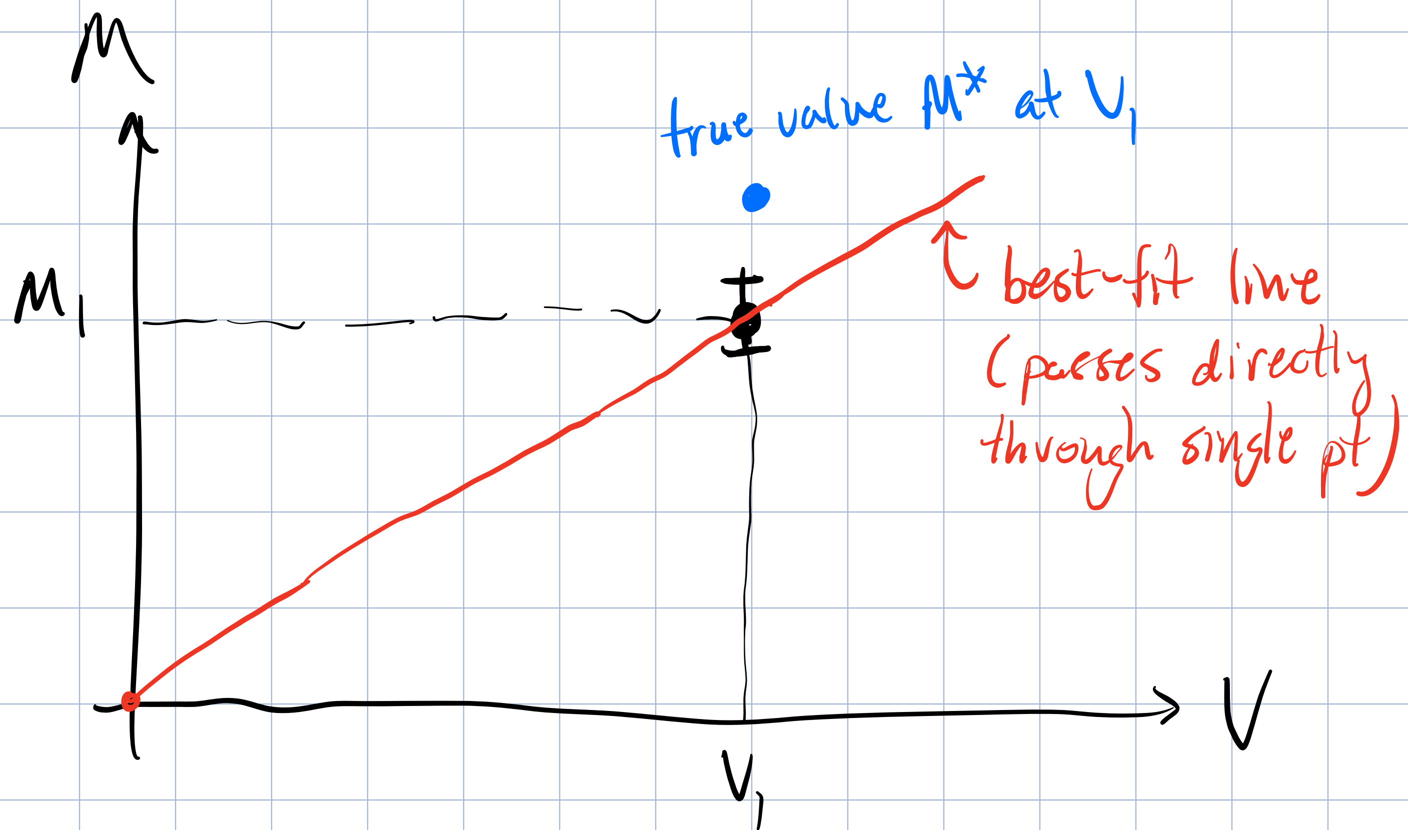
Meas. the volume of an object & measure its mass. Then, determine the density ρ from a fit to the data.

$$M = \rho V$$

y, slope, x

M vs V has a slope $b = \rho$ & intercept of zero.

1-parameter fit (p) w/ $N=1$ data pts.



If we have only a single pt to fit to, the deviation $y_i - y(x_i) = 0$ & $\therefore \chi^2 = 0$.

Since our best-fit line is biased to pass exactly through our single measurement, the calculate χ^2 value is artificially low ($\chi^2 = 0$).

If we could calculate χ^2 using the true value of

M (called M^* in figure) then we would get $\chi^2 \approx 1$ $(= N)$.

Resolution is that expected value of χ^2 is not N, but is $N - m$ where m is the number of parameters in your fit.

For $N=1$ & a one-parameter fit (slope or density ρ) we expect

$$\chi^2 = N - m = 1 - 1 = 0 \checkmark.$$

no. of fit parameters.

no. of data pts.

χ^2 summary:

- Expect $\chi^2 \approx N - m$

usually call $N - m = \nu$

"no. of degrees of freedom"

- If we make reasonable estimates of σ_i, then $\chi^2 > N-m$ indicates a poor fit.

- If $\chi^2 < N-m$, then prob. over-estimate the value of σ_i

Reduced χ^2 is denoted

$$\boxed{\chi_\nu^2 = \frac{\chi^2}{\nu} = \frac{\chi^2}{N-m}}$$

Expect $\chi_\nu^2 = 1$ for a good fit.

$\chi_\nu^2 > 1$ is an indication of a poor fit.