PHYS 232 March 1, 2024

Last Time:

Prob. of meas. dataset:

$$(x_1 \pm \sigma,\ x_2 \pm \sigma,\ \cdots,\ x_N + \sigma)$$

is given by

$$P = \prod_{i=1}^{N} P_i(x_i)\, dx$$

$$= \left(\frac{dx}{\sigma\sqrt{2\pi}}\right)^N \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i - \mu'}{\sigma}\right)^2\right]$$

To find the best estimate of μ', maximize P or, equivalently, minimize $X = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma}\right)^2$ w.r.t. μ'.

$\Rightarrow$ "Method of Maximum Likelihood"

Found $\mu' = \frac{1}{N} \sum_{i=1}^{N} x_i$, as expected.

Using propagation of errors, found error in our estimate of the mean was given by

$$\sigma_\mu = \frac{\sigma}{\sqrt{N}}$$

σ ← Standard deviation (error in any single measurement)

σ_μ ← standard error (error in the mean)

Weighted Mean

How should we combine diff. meas. $x_1, x_2, \ldots, x_N$ when each meas. has a different uncertainty $\sigma_1, \sigma_2, \ldots, \sigma_N$?

Eg. In PHYS 232 Lab, meas. Boltzmann's const. at three diff. temps & find

$$k_{B1} \pm \sigma_1, \quad k_{B2} \pm \sigma_2, \quad k_{B3} \pm \sigma_3$$

What is the appropriate value of $\overline{k_B}$ to report, what is σ_{k_B}?

$\Rightarrow$ Expect meas. w/ small uncertainties to be more important when determining average.

Again, approach is to use method of maximum likelihood $\Rightarrow$ maximize $P(\mu')$.

$$x_1 \pm \sigma_1 \,,\quad x_2 \pm \sigma_2 \,, \dots ,\; x_N \pm \sigma_N$$

$$P_1 = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left[-\frac{1}{2}\left(\frac{x_1-\mu'}{\sigma_1}\right)^2\right] dx$$

$$P(\mu') = \prod_{i=1}^{N} P_i$$

$$= \underbrace{\left[\prod_{i=1}^{n}\left(\frac{dx}{\sqrt{2\pi}\,\sigma_i}\right)\right]}_{\text{const. w.r.t. } \mu'} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma_i}\right)^2\right]$$

To maximize $P(\mu')$, minimize

$$\chi = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma_i}\right)^2 \quad \text{w.r.t. } \mu'$$

$$\frac{\partial \chi}{\partial \mu'} = \frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial \mu'}\left(\frac{x_i-\mu'}{\sigma_i}\right)^2 = 0$$

$$= \sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma_i}\right)\left(-\frac{1}{\sigma_i}\right) = 0$$

$$\therefore -\sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} + \mu'\sum_{i=1}^{N}\frac{1}{\sigma_i^2} = 0$$

solve for μ':

$$\mu'\sum_{i=1}^{N}\frac{1}{\sigma_i^2} = \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2}$$

divide by left-hand side sum

$$\boxed{\text{Weighted Mean:} \quad \mu' = \frac{\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}}$$

If all σ_i are equal : $\sigma_i = \sigma$

$$\mu' = \frac{\cancel{\frac{1}{\sigma^2}} \sum_{i=1}^{N} x_i}{\cancel{\frac{1}{\sigma^2}} \underbrace{\sum_{i=1}^{N} 1}_{N}} = \frac{1}{N} \sum_{i=1}^{N} x_i \quad \checkmark$$

(as expected)

Silly example

x_1 : $1 \pm \sigma$

x_2 : $1 \pm \sigma$

x_3 : $1 \pm \sigma$

x_4 : $2 \pm 2\sigma$

x_5 : $2 \pm 2\sigma$

x_6 : $2 \pm 2\sigma$

A "normal" average using $\frac{1}{N}\sum x_i$

would result in $\mu' = 1.5$

Expect weighted average to give a result closer to values w/ small uncertainty.

$$\mu' = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} = \frac{\frac{1}{\sigma^2} + \frac{1}{\sigma^2} + \frac{1}{\sigma^2} + \frac{2}{4\sigma^2} + \frac{2}{4\sigma^2} + \frac{2}{4\sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma^2} + \frac{1}{\sigma^2} + \frac{1}{4\sigma^2} + \frac{1}{4\sigma^2} + \frac{1}{4\sigma^2}}$$

$$= \frac{3 + \frac{3}{2}}{3 + \frac{3}{4}} = \frac{9/2}{15/4}$$

$$= \frac{9}{2}\,\frac{4}{15} = \frac{18}{15} = \frac{6}{5} = 1.2$$

As expected, our weighted is closer to 1 than standard mean of 1.5.

To find the uncertainty in the weighted mean, use propagation of errors.

$$\mu' = \frac{\sum_i \frac{x_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}}$$

$$\sigma_{\mu'}^2 = \left(\frac{\partial \mu'}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu'}{\partial x_2}\sigma_2\right)^2 + \dots + \left(\frac{\partial \mu'}{\partial x_N}\sigma_N\right)^2$$

Consider $\frac{\partial}{\partial x_j}(\mu') = \frac{\sum_i \frac{\partial}{\partial x_j}\left(\frac{x_i}{\sigma_i^2}\right)}{\sum_i \frac{1}{\sigma_i^2}}$

$j = 1 \dots N$

$$\frac{\partial \mu'}{\partial x_j} = \frac{\sum_i \frac{1}{\sigma_i^2} \frac{\partial x_i}{\partial x_j}}{\sum_i \frac{1}{\sigma_i^2}} \qquad \circledast$$

Consider $\frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Kroenker Delta

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise } (i \neq j) \end{cases}$$

Will will use the Kroenker delta as a "sum killer"

Eg. $$\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial x_i}{\partial x_j} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \delta_{ij}$$

$i=j$ term

$$= \frac{1}{\sigma_1^2} \cancelto{0}{\delta_{1j}} + \frac{1}{\sigma_2^2} \cancelto{0}{\delta_{2j}} + \dots + \frac{1}{\sigma_j^2} \cancelto{1}{\delta_{jj}} + \dots$$

$$+ \frac{1}{\sigma_N^2} \cancel{\delta_{Nj}} \to 0$$

$$\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial x_i}{\partial x_j} = \frac{1}{\sigma_j^2}$$

For any sum of the form

$$\sum_{i=1}^{N} f(y_i) \underbrace{\frac{\partial x_i}{\partial x_j}}_{\to\, \delta_{ij}} = f(y_j)$$

selects the $i=j$
term from sum

Returning to ⊛

$$\frac{\partial \mu'}{\partial x_j} = \frac{\frac{1}{\sigma_j^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}$$

$$\sigma_{\mu'}^2 = \left(\frac{\frac{1}{\sigma_1^{\cancel{2}}}}{\sum_i \frac{1}{\sigma_i^2}} \cancel{\sigma_1} \right)^2 + \left(\frac{\frac{1}{\sigma_2^{\cancel{2}}} \cancel{\sigma_2}}{\sum_i \frac{1}{\sigma_i^2}} \right)^2$$

$$+ \dots + \left(\frac{\frac{1}{\sigma_N^{\cancel{2}}} \cancel{\sigma_N}}{\sum_i \frac{1}{\sigma_i^2}} \right)^2$$

$$\sigma_{\mu'}^2 = \frac{1}{\left(\sum_i \frac{1}{\sigma_i^2} \right)^{\cancel{2}}} \cancel{\left[\underbrace{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \dots + \frac{1}{\sigma_N^2}}_{\sum_i \frac{1}{\sigma_i^2}} \right]}$$

$$\boxed{\therefore \quad \sigma_{\mu'}^2 = \frac{1}{\sum_{i=1} \frac{1}{\sigma_i^2}}}$$

uncertainty in the weighted mean.

If all σ_i are equal to σ

$$\sigma_{\mu'}^2 = \frac{1}{\sum_{i=1}^{N} \frac{1}{\sigma^2}} = \frac{1}{\frac{1}{\sigma^2} \underbrace{\sum_{i=1}^{N} 1}_{N}}$$

$$\sigma_{\mu'}^2 = \frac{1}{\frac{N}{\sigma^2}} = \frac{\sigma^2}{N}$$

$$\boxed{\therefore \sigma_{\mu'} = \frac{\sigma}{\sqrt{N}} \quad \text{expected standard error.}}$$