PHYS 232 Feb. 28, 2024

- Assignment # 3 due Friday
- Complete Lab # 4 sign up today

Last Time:

Propagation of Errors.

If $y = f(x_1, x_2, \ldots x_N)$

& meas. $x_1 \pm \sigma_{x_1}$

$x_2 \pm \sigma_{x_2}$

$\vdots$

$x_N \pm \sigma_{x_N}$

$$\boxed{\sigma_y = \sqrt{\left(\sigma_{x_1}\frac{\partial y}{\partial x_1}\right)^2 + \left(\sigma_{x_2}\frac{\partial y}{\partial x_2}\right)^2 + \ldots + \left(\sigma_{x_N}\frac{\partial y}{\partial x_N}\right)^2}}$$

Example: Boltzmann's constant lab

$$m = \frac{e}{k_B T} \Rightarrow k_B = \frac{e}{mT}$$

meas $m \pm \sigma_m$ & $T \pm \sigma_T$

& know $e \pm \sigma_e$.

Find $\sigma_{k_B}$.

Apply prop. of errors:

$$\sigma_{k_B}^2 = \left(\frac{\partial k_B}{\partial m}\sigma_m\right)^2 + \left(\frac{\partial k_B}{\partial T}\sigma_T\right)^2 + \left(\frac{\partial k_B}{\partial e}\sigma_e\right)^2$$

$$\frac{\partial k_B}{\partial m} = \frac{\partial}{\partial m}\left(\frac{e}{mT}\right) = -\frac{e}{m^2T}$$

$$\frac{\partial k_B}{\partial T} = -\frac{e}{mT^2}$$

$$\frac{\partial k_B}{\partial e} = \frac{1}{mT}$$

$$\sigma_{k_B}^2 = \left(\frac{e}{m^2T}\sigma_m\right)^2 + \left(\frac{e}{mT^2}\sigma_T\right)^2 + \left(\frac{\sigma_e}{mT}\right)^2 \quad (1)$$

- Sub in known / measured values to find $\sigma_{k_B}$.

- no rules to remember, just always apply general prop. of errors expression.

- prop. of errors works for <u>any</u> differentiable fcn.

Eg. $y = \ln x$

meas. $x \pm \sigma_x$. what's $\sigma_y$?

$$\sigma_y^2 = \left(\frac{\partial y}{\partial x}\sigma_x\right)^2 \Rightarrow \sigma_y = \left|\frac{\partial y}{\partial x}\sigma_x\right|$$

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$$

$$\boxed{\sigma_y = \left|\frac{\sigma_x}{x}\right|}$$

If we did error analysis with $k_B = \dfrac{e}{mT}$

using first-year rules we would:

$$\frac{\sigma_{k_B}}{k_B} = \frac{\sigma_e}{e} + \frac{\sigma_m}{m} + \frac{\sigma_T}{T}$$

$$\sigma_{k_B} = k_B \left[ \frac{\sigma_e}{e} + \frac{\sigma_m}{m} + \frac{\sigma_T}{T} \right]$$

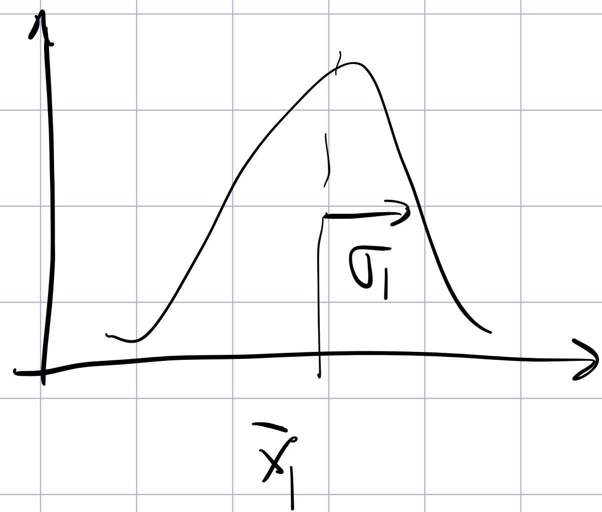
$$\frac{e}{mT}$$

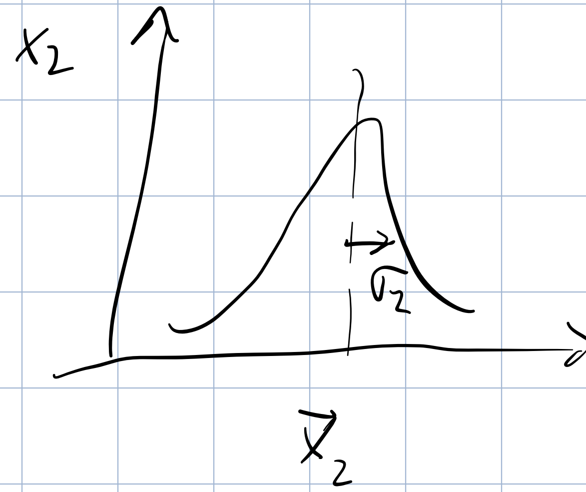
$$\sigma_{k_B} = \frac{\sigma_e}{mT} + \frac{e}{m^2 T}\sigma_m + \frac{e}{mT^2}\sigma_T \quad (2)$$

Eq'ns (1) & (2) have the same terms
→ differ only in $(\ )^2$ & $\sqrt{\ }$.

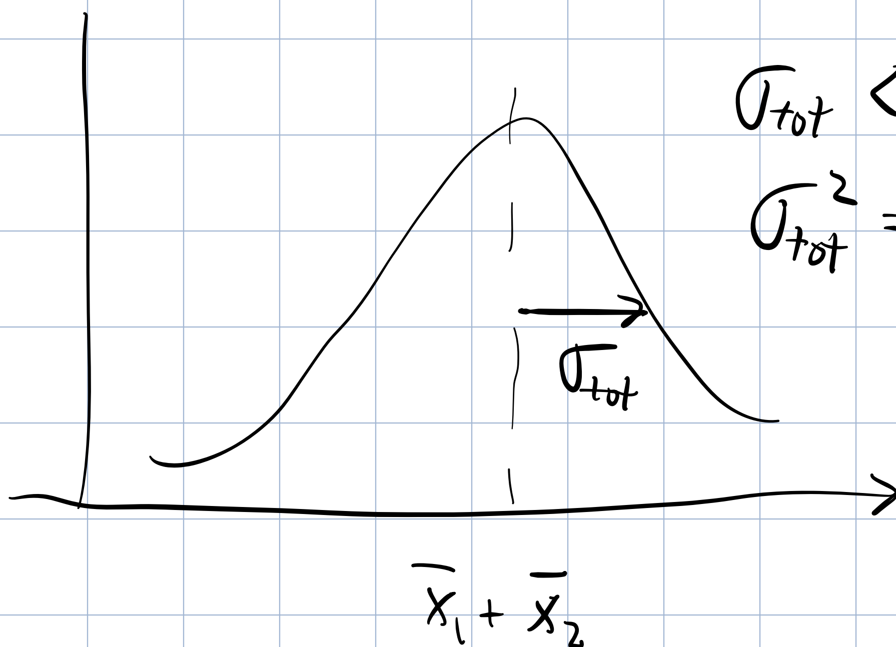
First-year result (2) over estimates the true uncertainty given by (1).

$x_1$




$x_1 + x_2$




$\sigma_{tot} < \sigma_1 + \sigma_2$

$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2$

Determine how to best estimate the mean from a set of N measurements.

Assume we've made N meas. of quantity $x$ & the data are Gaussian distributed.

Prob. that meas $i$ falls between $x_i$ & $x_i + dx$ is:

$$P_i = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right] dx$$

$\mu$ is mean of "parent" dist'n. Usually don't know its value ahead of time.

Our N meas. generate a "sample" dist'n with mean $\mu'$ & est. that prob of meas. $x_i$ with trial $i$ is:

$$P_i = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu'}{\sigma}\right)^2\right] dx$$

Total prob. of meas $x_1$ w/ trial 1
$x_2$ w/ trial 2
$\vdots$
$x_N$ w/ trial N

is

$$P(\mu') = P_1 P_2 \cdots P_N$$

$$= \prod_{i=1}^{N} P_i(\mu') = \underbrace{\left(\frac{dx}{\sigma\sqrt{2\pi}}\right)^N}_{\substack{\text{Const} \\ \text{w.r.t. } \mu'}} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i - \mu'}{\sigma}\right)^2\right]$$

"Method of Maximum Likihood", assumes that actual dataset & value of $\mu'$ give the max. value of $P(\mu')$.

Evaluate $\dfrac{\partial P(\mu')}{\partial \mu'} = 0$

solve for $\mu'$ to find value that maximizes $P(\mu')$.

Equivalently, we can minimize

$$X = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{x_i - \mu'}{\sigma} \right)^2$$ in the exponential to maximize $P(\mu')$

$$\frac{\partial X}{\partial \mu'} = \frac{\partial}{\partial \mu'}\left[\frac{1}{2}\sum_{i=1}^{N}\left(\frac{x_i-\mu'}{\sigma}\right)^2\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial \mu'}\left(\frac{x_i-\mu'}{\sigma}\right)^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} -2\,\frac{(x_i-\mu')}{\sigma^2} = 0$$

$$-\frac{1}{\cancel{\sigma^2}}\sum_{i=1}^{N} x_i + \frac{\mu'}{\cancel{\sigma^2}}\underbrace{\sum_{i=1}^{N} 1}_{N} = 0$$

$$\therefore -\sum_{i=1}^{N} x_i + \mu' N = 0$$

solve for $\mu'$:

Best est. of mean based off of N meas.

$$\boxed{\mu' = \frac{1}{N}\sum_{i=1}^{N} x_i}$$

Find the uncertainty in our est of the mean.

$\sigma_{\mu'}$ when $\mu'(x_1, x_2, \dots, x_N)$

Apply Prop. of errors.

$$\sigma_{\mu'}^2 = \left(\frac{\partial \mu'}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu'}{\partial x_2}\sigma_2\right)^2 + \dots \left(\frac{\partial \mu'}{\partial x_N}\sigma_N\right)^2$$

If we repeat the same kind of meas over & over, expect $\sigma_1 = \sigma_2 = \dots \sigma_N \equiv \sigma$

$$(\sigma_{\mu'})^2 = \sigma^2\left[\left(\frac{\partial \mu'}{\partial x_1}\right)^2 + \left(\frac{\partial \mu'}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial \mu'}{\partial x_N}\right)^2\right]$$

Consider $\frac{\partial}{\partial x_1}\mu' = \frac{\partial}{\partial x_1}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right]$

$$\frac{1}{N}\underbrace{\frac{\partial}{\partial x_1}\left(x_1 + x_2 + \ldots + x_N\right)}_{1}$$

$$\therefore \frac{\partial \mu'}{\partial x_1} = \frac{1}{N}$$

$$\frac{\partial \mu'}{\partial x_2} = \frac{1}{N}$$

$$\vdots$$

$$\frac{\partial \mu'}{\partial x_N} = \frac{1}{N}$$

$$\sigma_{\mu'}^{2} = \sigma^{2}\left[\left(\frac{1}{N}\right)^2 + \left(\frac{1}{N}\right)^2 + \ldots + \left(\frac{1}{N}\right)^2\right]$$

$$= \sigma^2 \left[ N\left(\frac{1}{N^2}\right) \right] = \frac{\sigma^2}{N}$$

$$\boxed{\therefore\ \sigma_{\mu} = \frac{\sigma}{\sqrt{N}}}$$

std dev. (uncertainty in any single meas.).

std. error or error in the mean