PHYS 232 Feb. 9, 2024

Last Time: Poisson Distribution.

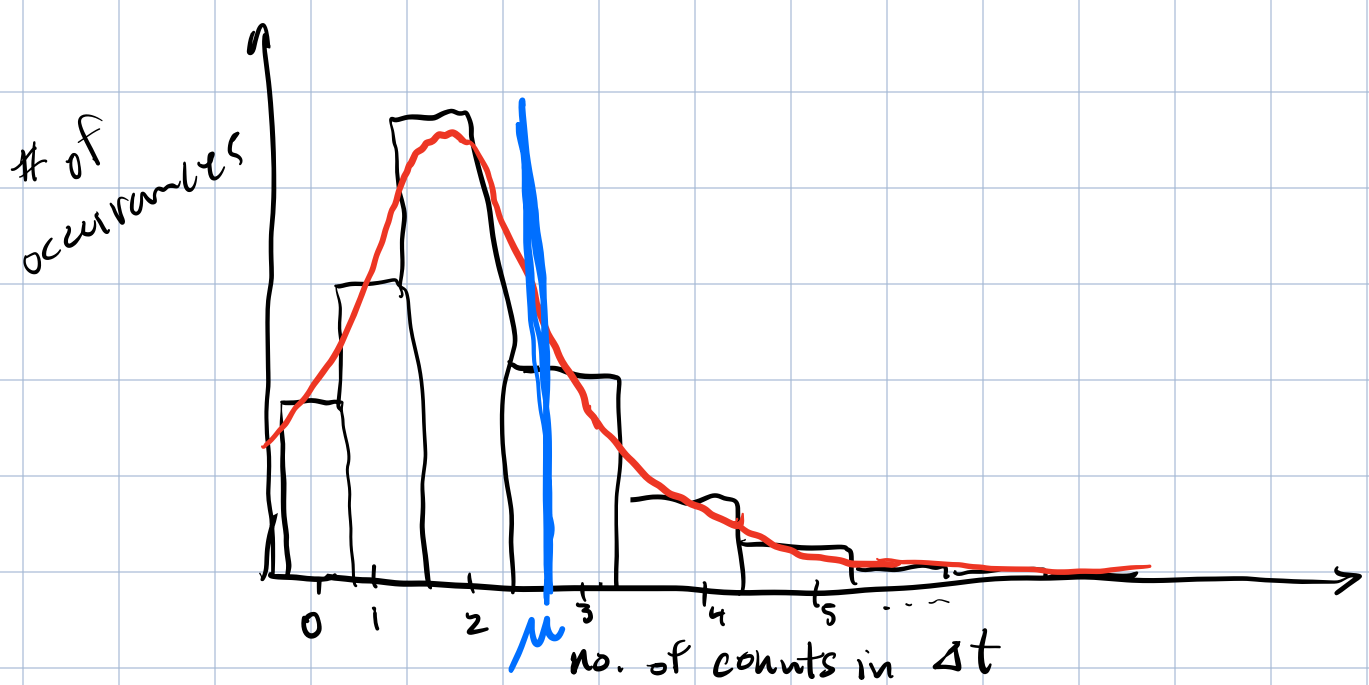
$$P_p(x;\mu) = \frac{\mu^x}{x!}e^{-\mu}$$

} $p \ll 1$ limit of the binomial dist'n.

$$\langle x \rangle = \mu$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\mu}$$

Eg. Count no. of beta decays from a sample in time Δt.



If we want to know prob. of getting between x_1 & x_2 counts during Δt.

$$P_{x_1 - x_2} = P_p(x_1; \mu) + P_p(x_1+1; \mu)$$

$$+ P_p(x_1+2; \mu) + \dots + P_p(x_2 - 1, \mu)$$

$$+ P_p(x_2, \mu).$$

$$= \sum_{x=x_1}^{x_2} P_p(x; \mu).$$

<u>Examples</u> Avg. no. of flaws in a fibre optic cable of length 50m is 1.2.

(i) What is prob. of exactly 3 flaws in 150m of cable?

$$P_p = \frac{\mu^x}{x!} e^{-\mu}$$

For 150m of cable, expect avg. no of flaws to be $\mu = 3(1.2) = 3.6$.

If we require 3 flaws, $x = 3$.

$$P_P(x; \mu) = \frac{(3.6)^3}{3!} e^{-3.6} = \underline{\underline{0.212}}$$

Note that P_P is a limit of P_B, so we should get a similar when using binomial dist'n.

$$P_B = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$n = 150$ (each 1m length is like a trial)
$x = 3$ (looking for prob. of 3 flaws)

$p = \frac{1.2}{50} = 0.024$ prob. of a flaw in 1m of cable.

$$P_B(3; 150, 0.024) = \frac{150!}{3!(147)!}\left(\frac{1.2}{50}\right)^3\left(1-\frac{1.2}{50}\right)^{150-3}$$

$$= \underline{\underline{0.214}}$$

(ii) What is prob. of at least two flaws in 100m of cable?

Prob. of getting $x = 2, 3, 4, 5, 6, \ldots, \infty$ flaws

$= 1 -$ (prob. of getting 0 or 1 flaw)

$$1 - P_P(0, \mu) - P_P(1, \mu)$$

$$\mu = 2(1.2) = 2.4$$

$$1 - 0.091 - 0.218 = \boxed{0.693}$$

Using Binomial dist'n.

$p = 0.024$ like in (i).

$$P_B(1; 100, p) + P_B(0; 100, p)$$

$$\frac{100!}{1!\,(99)!} p^1 (1-p)^{99} + \frac{100!}{0!\,100!} p^0 (1-p)^{100}$$

$$= 100\,p(1-p)^{99} + (1-p)^{100}$$

$$= 0.217 + 0.088 = 0.305$$

$$1 - .305 = \boxed{0.695}$$

(iii) What is prob. of exactly on flaw in first 50 m of a cable $\boxed{\&}$ exactly one flaw in 2nd 50 m?

$\mu = 1.2$
$x = 1$

$$P_p(1, 1.2) = 0.361$$ Prob. of 1 flaw in 50 m length.

$$P = P_p(1, 1.2) \cdot P_p(1, 1.2)$$
$$= \boxed{0.131}$$

Binomial distn.

$$P_B(1; 50, p) \cdot P_B(1; 50, p)$$

$$= \left(\frac{50!}{1!\,49!} p^1 (1-p)^{49} \right)^2$$

$$= \left[50p(1-p)^{49} \right]^2 = \boxed{0.133}$$

"Derivation" of Gaussian Dist'n.

Recall the random walk problem, the prob. of taking x of n steps to right is given by binomial dist'n.

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

know $\mu = np$

$\sigma^2 = np(1-p)$

Take $p = \frac{1}{2}$ & consider the prob. of taking m more steps to right than to left.

i.e. $$x = \frac{n}{2} + \frac{m}{2}$$

$$n - x = \frac{n}{2} - \frac{m}{2}$$

no. of extra steps right $\quad x-(n-x)$

$$=\left[\cancel{\frac{n}{2}}+\frac{m}{2}\right]-\left[\cancel{\frac{n}{2}}-\frac{m}{2}\right]=m \checkmark$$

For $p=\frac{1}{2}$ $\quad \mu=\frac{n}{2}$

$$\sigma^2=\frac{n}{4}$$

$$p^x(1-p)^{n-x}=\left(\frac{1}{2}\right)^n$$

Prob. of m extra step right:

$$* \quad P(m,n)=\frac{n!}{\left(\frac{n}{2}+\frac{m}{2}\right)!\left(\frac{n}{2}-\frac{m}{2}\right)!}\left(\frac{1}{2}\right)^n$$

To make the factorials easier to work mathematically use Stirling's approx.

$$z! \simeq \sqrt{2\pi}\, z^{z} e^{-z} \sqrt{z} \quad \text{for } z \gg 1$$

eg. $10! = \underbrace{3.6288 \times 10^{6}}_{\text{exact}} \simeq \underbrace{3.5987 \times 10^{6}}_{\text{Stirling approx}}$

diff. 0.83%.

Assume that $m \ll n$ s.t. all factorials in $P(m,n)$ are large nos. & we can use Stirling's approx.

$$P(m,n) = \frac{\cancel{\sqrt{2\pi}}\, n^{n}\, \cancel{e^{-n}} \sqrt{n} \quad 2^{-n}}{\left(\cancel{\sqrt{2\pi}} \left(\frac{n}{2}+\frac{m}{2}\right)^{\left(\frac{n}{2}+\frac{m}{2}\right)} \cancel{e^{-\frac{n}{2}-\frac{m}{2}}} \sqrt{\frac{n}{2}+\frac{m}{2}}\right) \cdot \left(\sqrt{2\pi} \left(\frac{n}{2}-\frac{m}{2}\right)^{\left(\frac{n}{2}-\frac{m}{2}\right)} \cancel{e^{-\frac{n}{2}+\frac{m}{2}}} \sqrt{\frac{n}{2}-\frac{m}{2}}\right)}$$