PHYS 232 Feb. 7, 2024

Last Time:

<u>Binomial Dist'n:</u>

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Today: - Watch simulation of particles suspended in a solution settling due to gravity

- The Poisson dist'n.

The poisson dist'n is a special case of the binomial dist'n.

$\mu \ll n.$ since $\mu = np$ for binomial dist'n.

mean # of trials

$np \ll n$

or $\boxed{p \ll 1}$

Binomial dist'n approaches Poisson dist'n when the prob. of success $p \ll 1$ is small.

Eg. Counting experiments.

Count no. of muon decays in time interval Δt. Repeat many times. Usually get no decays for short Δt. $\Rightarrow$ Prob. p for observing a muon decay is small.

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{1}{x!} \underbrace{\overbrace{\frac{n!}{(n-x)!} p^x}^{\mu^x}}_{①} \underbrace{\overbrace{(1-p)^{-x}}^{1}}_{②} \underbrace{\overbrace{(1-p)^n}^{e^{-\mu}}}_{③}$$

① $$\frac{n!}{(n-x)!} = \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)}!}{\cancel{(n-x)}!}$$

Since $p \ll 1$, $x \ll n$ (few successes)

$$\therefore \left.\begin{aligned} (n-1) &\approx n \\ (n-2) &\approx n \\ &\vdots \\ (n-x+1) &\approx n \end{aligned}\right\} \text{ } x \text{ of these factors.}$$

$$\therefore \frac{n!}{(n-x)!} \approx n^x$$

$$\boxed{\therefore \underbrace{\frac{n!}{(n-x)!}}_{n^x} p^x = (np)^x = \mu^x}$$

② $(1-p)^{-x} \approx (1)^{-x} = 1$

③ $(1-p)^n$ know $pn = \mu$

$$\therefore n = \frac{\mu}{p}$$

$$(1-p)^n \approx (1-p)^{\mu/p}$$

$$= \left[(1-p)^{\frac{1}{p}}\right]^{\mu}$$

Claim: $\lim_{p\to 0}\left[(1-p)^{1/p}\right] = \frac{1}{e}$

Euler's no. 2.718..

Proof:

$$y = (1-p)^{1/p}$$

$$\ln y = \frac{1}{p}\ln(1-p)$$

$$\lim_{p\to 0} \ln y \approx \frac{1}{p}(-p)$$

$$= -1$$

Recall $\ln(1+x) \approx x$ when $x \ll 1$. (using Taylor series)

$\therefore y \approx e^{-1}$ when $p \ll 1$.

$$\boxed{\therefore (1-p)^{\frac{1}{p}} \approx \frac{1}{e} \text{ when } p \ll 1}$$

$$\therefore (1-p)^n = \left[(1-p)^{1/p}\right]^{\mu}$$

$$\approx \left[\frac{1}{e}\right]^{\mu} = e^{-\mu} \quad \blacksquare$$

$$\lim_{p \to 0} P_B(x; n, p) \simeq \frac{1}{x!} \mu^x e^{-\mu}$$

$$\boxed{\therefore \quad P_p(x; \mu) = \frac{\mu^x}{x!} e^{-\mu}}$$

Poisson Dist'n, $p \to 0$ limit of Binomial dist'n.

Check the requirement $\displaystyle\sum_{x=0}^{\infty} P_p = 1$.

$$\sum_{x=0}^{\infty} P_p(x; \mu) = \sum_{x=0}^{\infty} \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

$$= e^{-\mu} \underbrace{\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \ldots \right)}_{\text{Taylor series of } e^{\mu}}$$

$$\sum_{x=0}^{\infty} P_p(x;\mu) = e^{-\mu} e^{\mu} = 1 \checkmark$$

<u>Mean of Poisson dist'n</u>

$$\langle x \rangle = \sum_{x=0}^{\infty} x \, P_p(x;\mu)$$

$$= \sum_{x=0}^{\infty} x \, \frac{\mu^x}{x!} e^{-\mu}$$

$$= 0 + e^{-\mu} \sum_{x=1}^{\infty} \frac{x}{x!} \mu^x$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \mu^x$$

$$= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{1}{(x-1)!} \mu^{x-1}$$

sub $y = x - 1$

$$\langle x \rangle = \mu e^{-\mu} \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} \mu^y}_{e^{\mu} \text{ (Taylor series)}}$$

$$\boxed{\langle x \rangle = \mu}$$ mean of the Poisson dist'n.

Standard Deviation

$$\sigma^2 = \langle (x-\mu)^2 \rangle$$

$$= \sum_{x=0}^{\infty} (x-\mu)^2 P_p(x;\mu)$$

$$= \underbrace{\sum_{x=0}^{\infty} x^2 P_p(x;\mu)}_{\langle x^2 \rangle} - \mu^2$$

HW#3 You will show that

$$\sigma^2 = \mu \implies \sigma = \sqrt{\mu}$$

Poisson distr'n is characterized by a single paramter μ.

mean: μ
std. dev: $\sqrt{\mu}$.