PHYS 121 Feb. 2, 2024

Binomial Dist'n:

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\mu = np \qquad \sigma^2 = np(1-p)$$

Today: More on the 1-D random walk.

Let's try to calc. the mean displacement $\langle \Delta x \rangle$ & the mean of the square of the displacement $\langle \Delta x^2 \rangle$.

If we take $x$ step right, displacement to right $\Delta x_R = x\,l$ where $l$ is avg. step size.

Displacement to left will be $\Delta x_L = (n-x)\,l$

Net displacement $\Delta x = \Delta x_R - \Delta x_L$

$$= (2x - n)\,l$$

$$(\Delta x)^2 = (4x^2 - 4nx + n^2)\,l^2$$

For a prob. dist $P(x_j)$

$$\langle x \rangle = \sum_{j=0}^{n} x_j P(x_j)$$

$$\langle x^2 \rangle = \sum_{j=0}^{n} x_j^2 P(x_j)$$

In general, any fcn $f(x_j)$ has an avg. of

$$\langle f(x) \rangle = \sum_{j=0}^{n} f(x_j) P(x_j)$$

Start w/ find $\langle \Delta x \rangle$:

$$\langle \Delta x \rangle = \sum_{x=0}^{n} \Delta x \, P_B(x; n, p)$$

$$= \sum_{x=0}^{n} (2x - n) \ell \, P_B$$

$$= 2\ell \underbrace{\sum_{x=0}^{n} x \, P_B}_{\mu = np} - n\ell \underbrace{\sum_{x=0}^{n} P_B}_{1}$$

$$\therefore \langle \Delta x \rangle = 2\ell np - n\ell$$

$$\boxed{\langle \Delta x \rangle = (2p - 1) n\ell}$$

$\langle \Delta x \rangle = 0$ if $p = \frac{1}{2}$, as expected.

Now let's find $\langle \Delta x^2 \rangle$

$$\langle \Delta x^2 \rangle = \sum_{x=0}^{n} \underbrace{(4x^2 - 4nx + n^2)\ell^2}_{\Delta x^2} P_B$$

$$= 4\ell^2 \sum_{x=0}^{n} x^2 P_B - 4n\ell^2 \underbrace{\sum_{x=0}^{n} x P_B}_{np} + n^2\ell^2 \underbrace{\sum_{x=0}^{n} P_B}_{1}$$

$$\langle \Delta x^2 \rangle = 4\ell^2 \sum_{x=0}^{n} x^2 P_B - 4n^2 p\ell^2 + n^2\ell^2$$

Focus on evaluating $\displaystyle\sum_{x=0}^{n} x^2 P_B = \cancelto{0}{0^2 P_B(x=0)} + \sum_{x=1}^{n} x^2 P_B$

$$\langle x^2 \rangle = \sum_{x=1}^{n} x^2 \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\frac{x^2}{x!} = \frac{x^{\cancel{2}}}{\cancel{x}(x-1)!} = \frac{x}{(x-1)!}$$

$$n!p^x = n(n-1)!\,p\,p^{x-1}$$

$$= np\,(n-1)!\,p^{x-1}$$

$$\therefore \langle x^2 \rangle = np \sum_{x=1}^{n} x \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$$

sub. $y = x-1 \quad \therefore \quad x = y+1$

$x=1, \quad y=0$

$x=n, \quad y=n-1$

$$\langle x^2 \rangle = np \sum_{y=0}^{n-1} (y+1) \frac{(n-1)!}{y!\,(n-1-y)!} p^{y} (1-p)^{n-1-y}$$

sub $m = n-1$

$$\langle x^2\rangle = np\sum_{y=0}^{m}(y+1)\underbrace{\frac{m!}{y!\,(m-y)!}\,p^{y}(1-p)^{m-y}}_{P_B(y;m,p)}$$

$$\langle x^2\rangle = np\sum_{y=0}^{m}(y+1)\,P_B(y;m,p)$$

$$= np\left[\underbrace{\sum_{y=0}^{m} y\,P_B(y;m,p)}_{\langle y\rangle = mp \\ = (n-1)p} + \underbrace{\sum_{y=0}^{m} P_B(y;m,p)}_{1}\right]$$

$$\langle x^2\rangle = np\left[(n-1)p+1\right]$$

$$\boxed{\langle x^2\rangle = n^2p^2 - np^2 + np}$$

$\therefore$ our result for $\langle \Delta x^2 \rangle$ is:

displacement

$$\langle \Delta x^2 \rangle = 4\ell^2 \underbrace{\sum_{x=0}^{n} x^2 P_B}_{\langle x^2 \rangle} - 4n^2 p \ell^2 + n^2 \ell^2$$

no. of step right.

$$\langle \Delta x^2 \rangle = 4\ell^2 \left( n^2 p^2 - n p^2 + n p \right) - 4\ell^2 n^2 p + 4\ell^2 \left( \frac{n^2}{4} \right)$$

$$\boxed{\therefore \langle \Delta x^2 \rangle = 4\ell^2 \left[ n^2 p^2 - n p^2 + n p - n^2 p + \frac{n^2}{4} \right]}$$

Consider the special case of $p = \frac{1}{2}$.

$$\langle \Delta x^2 \rangle = 4\ell^2 \left[ \cancel{\frac{n^2}{4}} - \frac{n}{4} + \frac{n}{2} - \cancel{\frac{n^2}{2}} + \cancel{\frac{n^2}{4}} \right]$$

$$\boxed{\langle \Delta x^2 \rangle = 4 l^2 \left( \frac{n}{4} \right) = n l^2}$$

The average of the square of the displacement is non-zero & prop. to $n$, the number of ~~st~~ steps take.

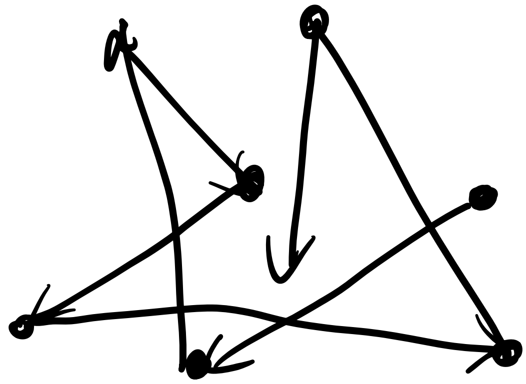
$$\sigma_{\Delta x} = \sqrt{\langle \Delta x^2 \rangle - \underbrace{\langle \Delta x \rangle^2}_{0 \text{ for } p = \frac{1}{2}}}$$

$$\sigma_{\Delta x} = \sqrt{n}\, l$$

If drunk need to take 10 step right to get home, on average will need to total of 100 steps to sample the required space.

Return to $\langle \Delta x^2 \rangle = n l^2$

This result also applies to the Brownian motion of a particle suspended in a fluid.



here $l$ is the avg. distance traveled by particle between scattering events. $l$ is called the "mean free path".

If average time between scattering events is $\tau$, then total time for $n$ scatters will be $t = n\tau$

$\therefore n = \frac{t}{\tau}$ ← total time.

$$\langle \Delta x^2 \rangle = \frac{t\, l^2}{\tau}$$

It is common to define a diffusion coefficient

$$D \equiv \frac{l^2}{2\tau}$$

$$\boxed{\langle \Delta x^2 \rangle = 2Dt}$$

average displacement of suspended particle in 1-D after a time $t$.

For a spherical particle suspend in a fluid, the diffusion coefficient can be calculated & it is given by:

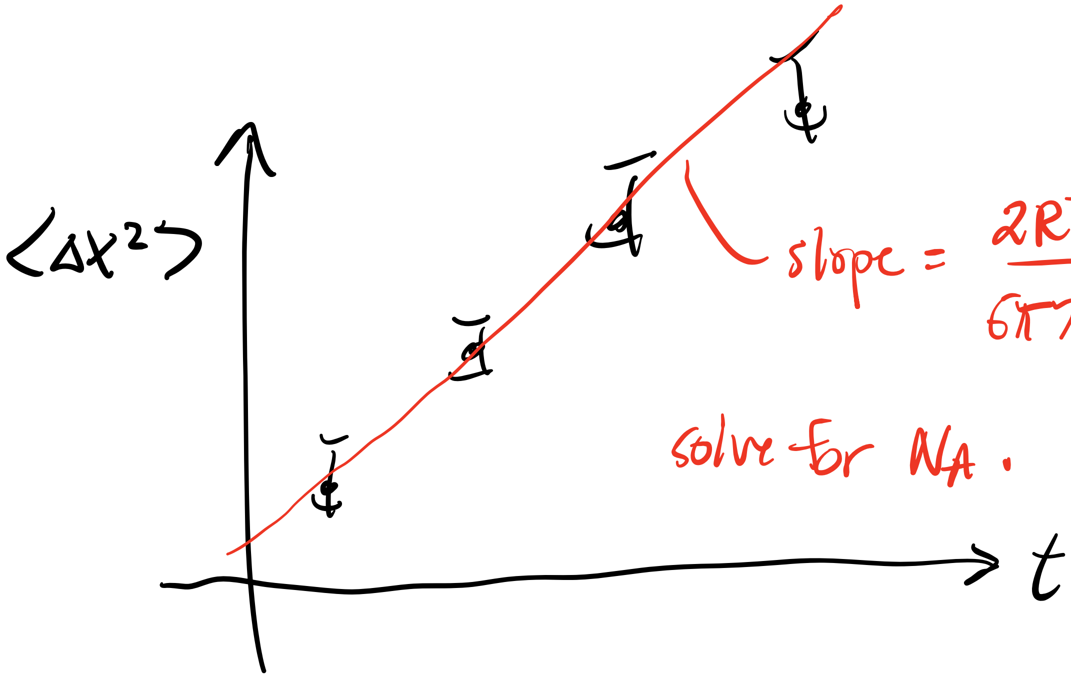
$$D = \frac{k_B T}{6\pi \eta a} = \frac{RT}{6\pi \eta N_A a}$$

$T$: temperature

$\eta$: viscosity of fluid

$a$: particle radius

$$\boxed{\langle \Delta x^2 \rangle = \left( \frac{2RT}{6\pi \eta N_A a} \right) t}$$




slope $= \dfrac{2RT}{6\pi \eta N_A a}$

solve for $N_A$.