PHYS 232 January 31, 2024

Last Time:

The binomial prob. distr:

$$P_B(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

binomial

no. of "successes"

no. of trials

prob. of any single trial being successful

$$\boxed{P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}}$$

Recall $\mu = \sum_{j=1}^{n} x_j P(x_j)$

$$\sigma^2 = \sum_{j=1}^{n} x_j^2 P(x_j) - \mu^2$$

To find μ of Binomial dist'n, have to sum over all possible outcomes when we have n trials.

Possible Outcomes

$x = 0$ (no successes in n trials)

$x = 1$ 1 success

$x = 2$ 2 successes

$\vdots$

$x = n$ n successes.

$$\mu = \underbrace{\sum_{x=0}^{n}}_{\text{sum over all possible outcomes.}} x \, P_B$$

$$\mu = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

expecting to find $\mu = np$ ← prob. of success for any single trial.

no. trials

- note that $\frac{x}{x!} = \frac{1}{(x-1)!}$
- write $\sum_{x=0}^{n} x(\quad) = \cancel{0(\quad)}^{\,0} + \sum_{x=1}^{n} x(\quad)$

$$\mu = \sum_{x=1}^{n} \cancel{x} \frac{n!}{\cancel{x}(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

write $p^x = p(p^{x-1})$

$n! = n(n-1)!$

$$\mu = \sum_{x=1}^{n} \boxed{np} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

$$\mu = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$$

make a substitution $y = x-1 \Rightarrow x = y+1$

sum limits:

when $x=1$, $y=0$

$x=n$, $y=n-1$

$$\mu = np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!\,(n-(y+1))!} p^{y} (1-p)^{n-(y+1)}$$

$$= np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!\,((n-1)-y)!} p^{y} (1-p)^{(n-1)-y}$$

another substitution: $m \equiv n-1$

$$\mu = np \underbrace{\sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y}}_{P_B(y;m,p)}$$

$$\sum_{y=0}^{m} P_B(y;m,p) = 1$$ summing over all possible outcomes of a prob dist'n.

$$\boxed{\therefore \mu = np}$$

If we were to calc. the standard dev.

$$\sigma^2 = \sum_{x=0}^{n} (x-\mu)^2 P_B(x;n,p) = np(1-p)$$

(HW2).

Eg. 1-D Random Walk.

Drunk starts at stop sign. Has prob. p of taking step to the right & prob. $q = 1-p$ of taking a step to left.

After n steps of size Δx, what is avg. dist. from sign? What is std. dev.?

Avg. no. of steps right: $\mu_R = np$

Avg. displacement right: $np\Delta x$

Avg. no. of steps left: $\mu_L = n(1-p)$

Avg. displacement left: $-n(1-p)\Delta x$

net displacement: $np\Delta x - n(1-p)\Delta x$

$$= np\Delta x - n\Delta x + np\Delta x$$

$$= \boxed{n\Delta x(2p-1)}$$

If $p = \frac{1}{2}$, net displacement is zero, as expected?

$\sigma^2 = np(1-p)$ spread in no. of steps to right increases as $\sqrt{n}$.

$$P_B = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

If $p = q = \frac{1}{2}$

$n = 10$

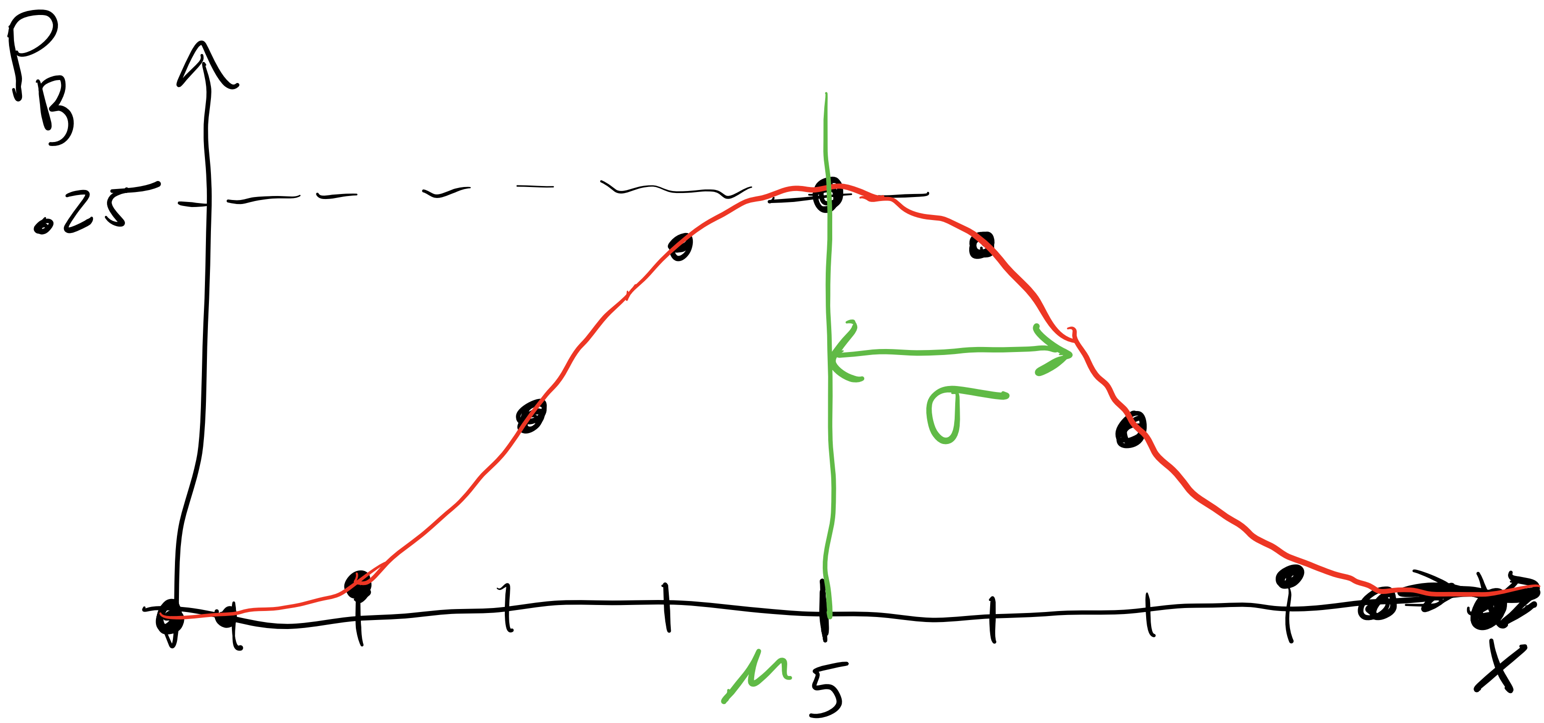
x	$P_B(x; 10, \frac{1}{2})$
0	9.8×10^{-4}
1	9.8×10^{-3}
2	4.4×10^{-2}
3	0.12
4	0.21
5	0.25
6	0.21
⋮	
10	9.8×10^{-4}

$\mu = np = 5$

$\sigma^2 = np(1-p)$

$$\sigma = \sqrt{\frac{5}{2}}$$

dist'n is symmetric when $p = q = \frac{1}{2}$.



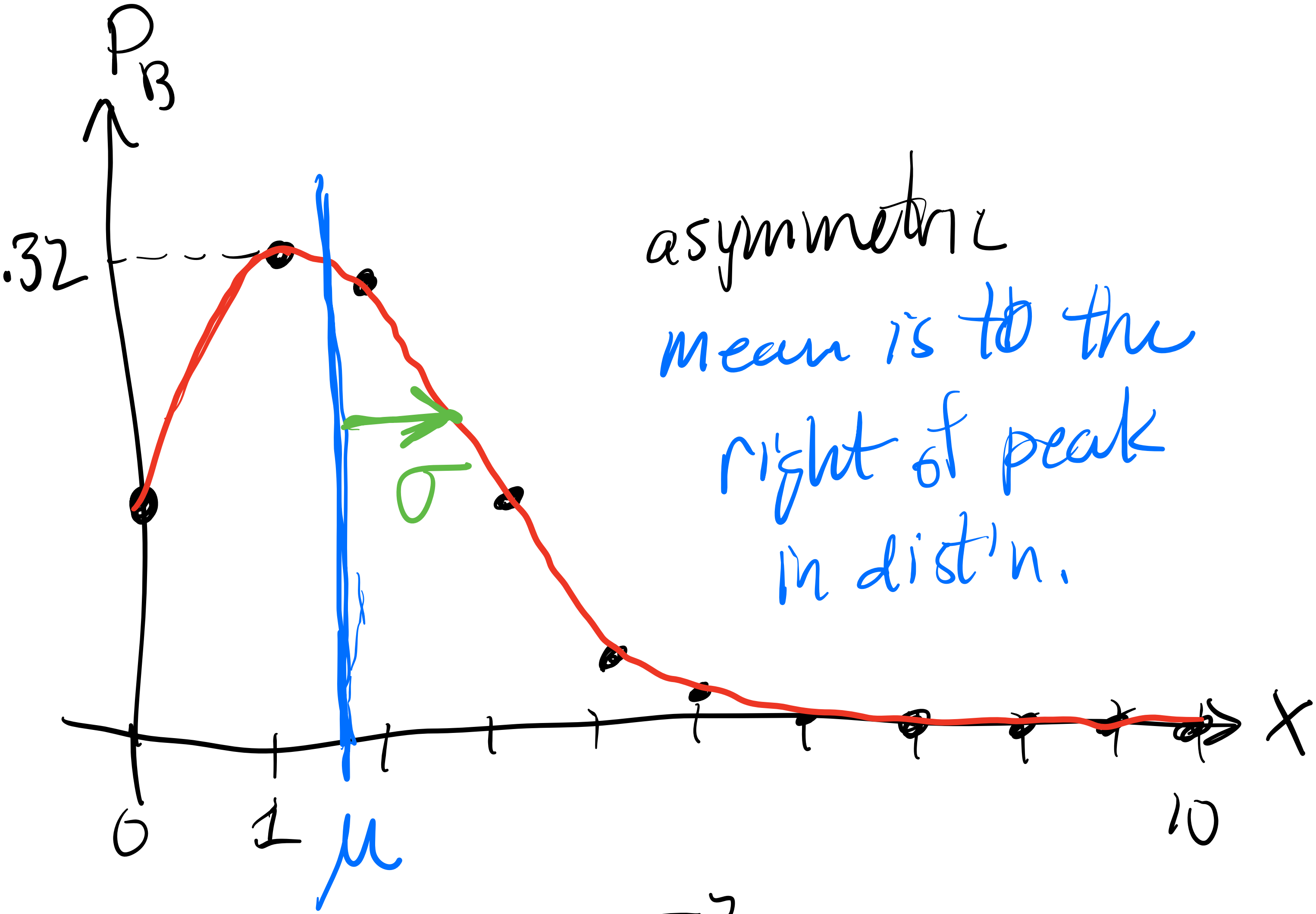
Try $p = \frac{1}{6}$ $\quad q = 1 - p = \frac{5}{6}$

x	$P_B(x; 10, \frac{1}{6})$
0	$\left(\frac{5}{6}\right)^{10} = 0.16$
1	.32
2	.29
3	.16
4	.054
5	.013
⋮	⋮
10	$\left(\frac{1}{6}\right)^{10} = 1.6 \times 10^{-8}$

$\mu = np$

$= \frac{10}{6} = \frac{5}{3}$

$= 1.67$



asymmetric

mean is to the right of peak in dist'n.

$$\sigma^2 = np(1-p)$$

$$= 10\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$$

$$= \frac{50}{36}$$

$$\sigma \approx \frac{7}{6}$$