PHYS 232 Jan. 26, 2024

Today: The Binomial distribution

→ An example of a discrete probability distribution

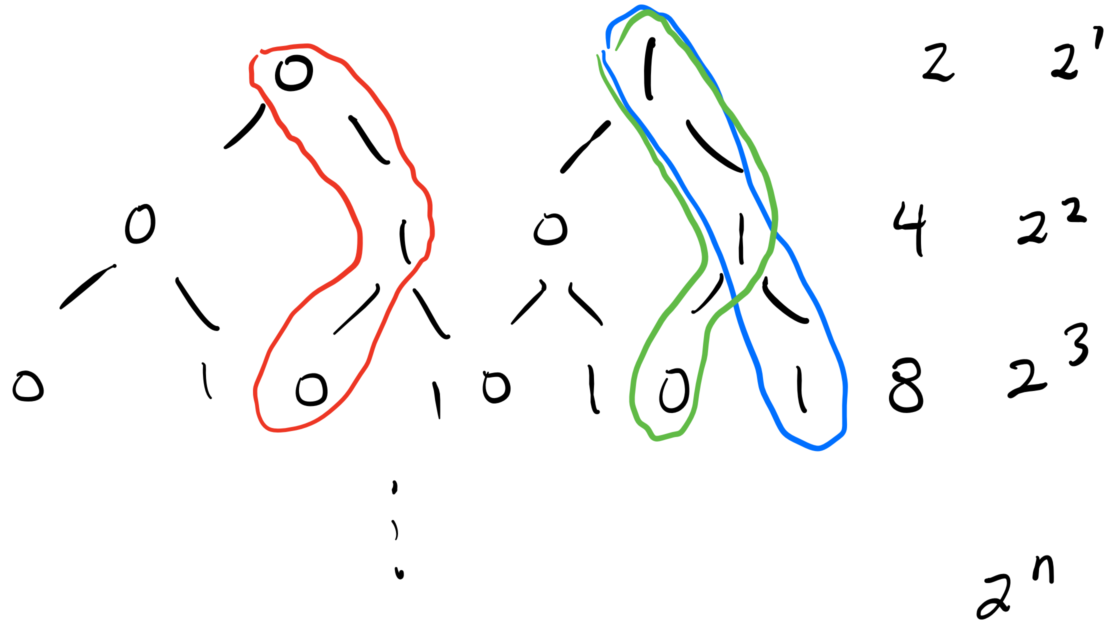
Use when the result of an experiment can be only one of a small no. of outcomes.

- rolling die
- flipping a coin
- drawing a card.

If toss n coins in air, how may possible outcomes are there.

First coin has 2 possible outcomes

2nd " " " " "



For n coins have a total of 2^n outcomes.

The prob. of each individual outcome are all equal to $\frac{1}{2^n}$.

For $n=5$ prob. of getting 00000 same as prob. of getting exactly 01100 in this order

coin 1 2 3 4 5

However, there are many ways of getting 2 heads & 3 tails.

11000 01100 00110
10100 01010 00101
10010 01001
10001
00011

10

In general, if we toss n coins, how many ways or <u>permutations</u> $P_m(n, x)$ are there of x heads & n−x tails when we do n trials?

① Select from n coins in front of you & put 1 into heads pile.

n choices

② Select from n−1 coins & add 1 to heads pile

n−1 choices

⋮

ⓧ Select from $n-(x-1)$ coins remaining
& add 1 into heads pile

$(n-x+1)$ choices.

Total no. of permutations is:

$$P_m(n,x) = n(n-1)(n-2)\cdots(n-x+2)(n-x+1)$$

$$= \frac{n!}{(n-x)!}$$

Eg. $n=5$, $x=2$

$$P_m(5,2) = \frac{5!}{3!} = 5\cdot 4 = \boxed{20}$$

Why is $P_m(5,2) = 20$ different than 10 possibilities we drew?

Consider the 11000 outcome.

$P_m(n,x)$ considers

$1_1 \quad 1_2 \quad 0 \quad 0 \quad 0$

$1_2 \quad 1_1 \quad 0 \quad 0 \quad 0$

as distinct possibilities

Choosing coin 1 first or second is distinct.

If have x heads, there are x! ways to order them in heads pile.

Eg. $x = 2$

$1_1 \quad 1_2$

$1_2 \quad 1_1$

$x! = 2! = 2$

$x = 3$

$1_1 \quad 1_2 \quad 1_3$

$1_3 \quad 1_1 \quad 1_2$

$1_2 \quad 1_3 \quad 1_1$

$1_3 \quad 1_2 \quad 1_1 \qquad 3! = 6$

$1_1 \quad 1_3 \quad 1_2$

$1_2 \quad 1_1 \quad 1_3$

The no. of combinations $C(n,x)$ of x heads from n coins when the order of x doesn't matter is

$$C(n,x) = \frac{P_m(n,x)}{x!} = \frac{n!}{x!\,(n-x)!}$$

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$ often read as "n choose x"

Eg. $n=5$, $x=2$

$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!}$$

$$= \frac{5 \cdot 4}{2} = 10 \checkmark$$

Probability

Since each outcome has the same prob $\left(\frac{1}{2^n}\right)$ of occurring, the prob of getting x heads from n coins is:

$$P(x;n) = \binom{n}{x}\frac{1}{2^n} = \underbrace{\frac{n!}{x!(n-x)!}}_{\substack{\text{no. of} \\ \text{combinations}}} \underbrace{\frac{1}{2^n}}_{\substack{\text{prob. of} \\ \text{any} \\ \text{particular} \\ \text{outcome.}}}$$

Valid only when prob. of getting result x (heads) in any single trial is $\frac{1}{2}$.

In general, the prob. of the possible outcomes could be different.

Eg. weighted coin.

prob. of getting heads $P_H = p$ (prob. of "success")

prob of getting tails $P_T = q$ (prob. of "failure")

Then the prob. of x heads & $n-x$ tails is

$$p^x q^{n-x} = p^x (1-p)^{n-x}$$

when $p+q=1$.

The binomial prob. distn is:

$$P_B(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

binomial

no. of "successes"

no. of trials

prob. of any single trial being successful

$$\boxed{P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}}$$

Next time: find the mean & std. dev. of binomial distn.

$$\mu = \sum_{j=1} x_j \, P(x_j)$$

$$\sigma^2 = \underbrace{\langle x^2 \rangle}_{\sum_j x_j^2 P(x_j)} - \underbrace{\langle x \rangle^2}_{\mu^2}$$