PHYS 231 Oct. 10, 2025

## Last Time:

If $z = x + jy = |z|e^{j\phi}$
then, the complex conjugate of $z$, denoted $z^*$, is $z^* = x - jy = |z|e^{-j\phi}$

When finding the complex conjugate, replace all instances of $j$ with $-j$.

Why is the complex conjugate a useful quantity?

To answer this question, we will consider $z_1^* z_2$ where:

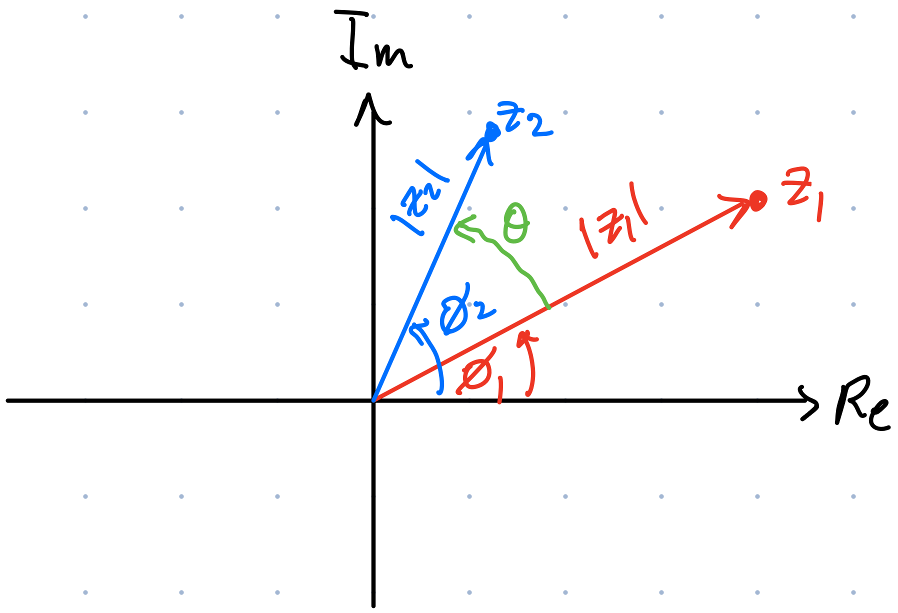
$$z_1 = x_1 + jy_1 = |z_1|e^{j\phi_1}$$
$$z_1^* = x_1 - jy_1 = |z_1|e^{-j\phi_1}$$

$$z_2 = x_2 + jy_2 = |z_2|e^{j\phi_2}$$

In magnitude & phase form:

$$z_1^* z_2 = |z_1| e^{-j\phi_1} |z_2| e^{j\phi_2}$$

$$= |z_1||z_2| e^{j(\phi_2 - \phi_1)}$$




$\theta = \phi_2 - \phi_1$

↑ angle between the pair of complex nos.

Using Euler's eq'n

$$z_1^* z_2 = |z_1||z_2| \left[ \cos(\underbrace{\phi_2 - \phi_1}_{\theta}) + j \sin(\underbrace{\phi_2 - \phi_1}_{\theta}) \right]$$

$$z_1^* z_2 = \underbrace{|z_1||z_2| \cos\theta}_{\text{like the dot product}} + j \underbrace{|z_1||z_2| \sin\theta}_{\text{like the cross product}}$$

If we had vectors $\vec{V}_1$ & $\vec{V}_2$, $\quad \vec{V}_1 \cdot \vec{V}_2 = |\vec{V}_1||\vec{V}_2| \cos\theta$

$$|\vec{V}_1 \times \vec{V}_2| = |\vec{V}_1||\vec{V}_2| \sin\theta$$

- If $\mathrm{Re}\left[z_1^* z_2\right] = 0$, then we know that

  $\theta = \pm\frac{\pi}{2}$ (angle between $z_1$ & $z_2$)

  $\Rightarrow z_1 \perp z_2$ in the complex plane.

- If $\mathrm{Im}\left[z_1^* z_2\right] = 0$, then we know that

  $\theta = 0, \pi$

  $\Rightarrow z_1$ and $z_2$ are either parallel ($\mathrm{Re}[z_1^* z_2]$ positive) or antiparallel ($\mathrm{Re}[z_1^* z_2]$ negative).

If we consider $z_1^* z_1 = |z_1| e^{-j\phi_1} \, |z_1| e^{j\phi_1}$

$$= |z_1|^2 e^{j(\phi_1 - \phi_1)}$$

$$\boxed{\therefore z_1^* z_1 = |z_1|^2}$$

Taking $z_1$ and multiplying by its own conjugate returns a completely real result that is equal to the square of the magnitude of $z_1$.

The other use of complex conjugates is for simplifying complex numbers.

Eg. $$z = \frac{6+j5}{3-j2} = \frac{z_1}{z_2}$$

To get $z$ in the form $z = x + jy$, we need to remove the $j$ from the denominator.

Strategy is to multiply $z$ by $1 = \frac{z_2^*}{z_2^*}$.

The $z_2 z_2^*$ product in the denominator will make $|z_2|^2$ (real).

$$z = \frac{(6+j5)}{(3-j2)} \frac{(3+j2)}{(3+j2)} = \frac{18+12j+15j-10}{9+4}$$

$$z = \frac{8+27j}{13}$$

$$x = \frac{8}{13} \quad y = \frac{27}{13}$$

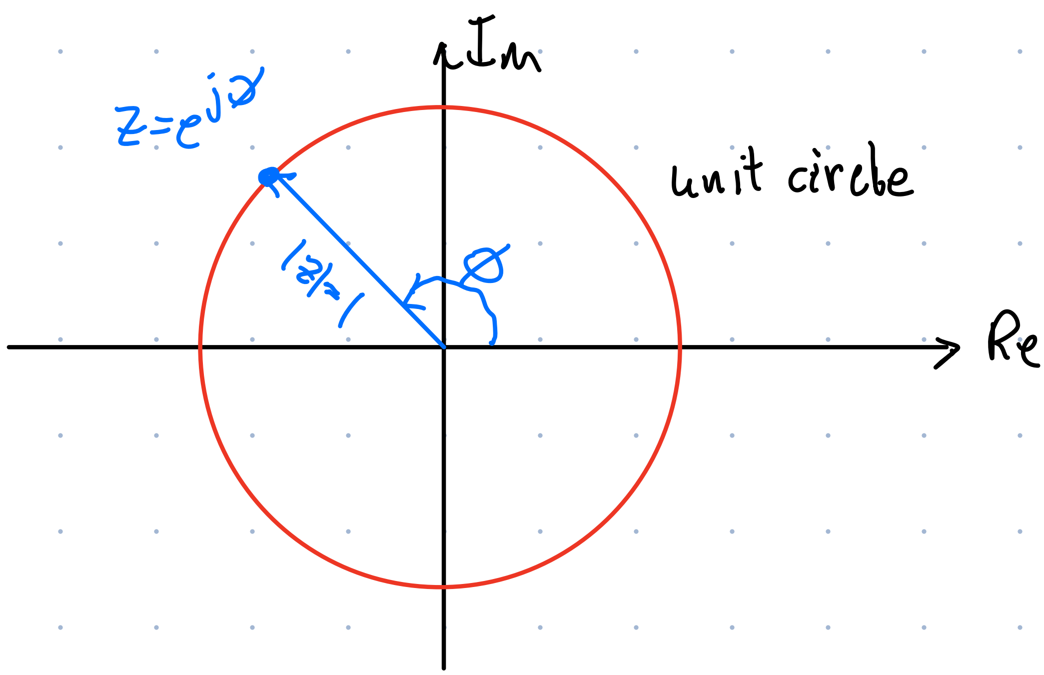
Return to Euler's Eq'n in order to develop some intution.

$$\underbrace{z = e^{j\phi}}_{\text{complex no. of length 1.}} = \cos\phi + j\sin\phi$$

What is $|z|$? $\quad |z|^2 = z^* z = e^{-j\phi} e^{j\phi} = 1$

$\therefore$ Distance from the origin to $z = e^{j\phi}$ in the complex plane is 1 for any value of $\phi$.

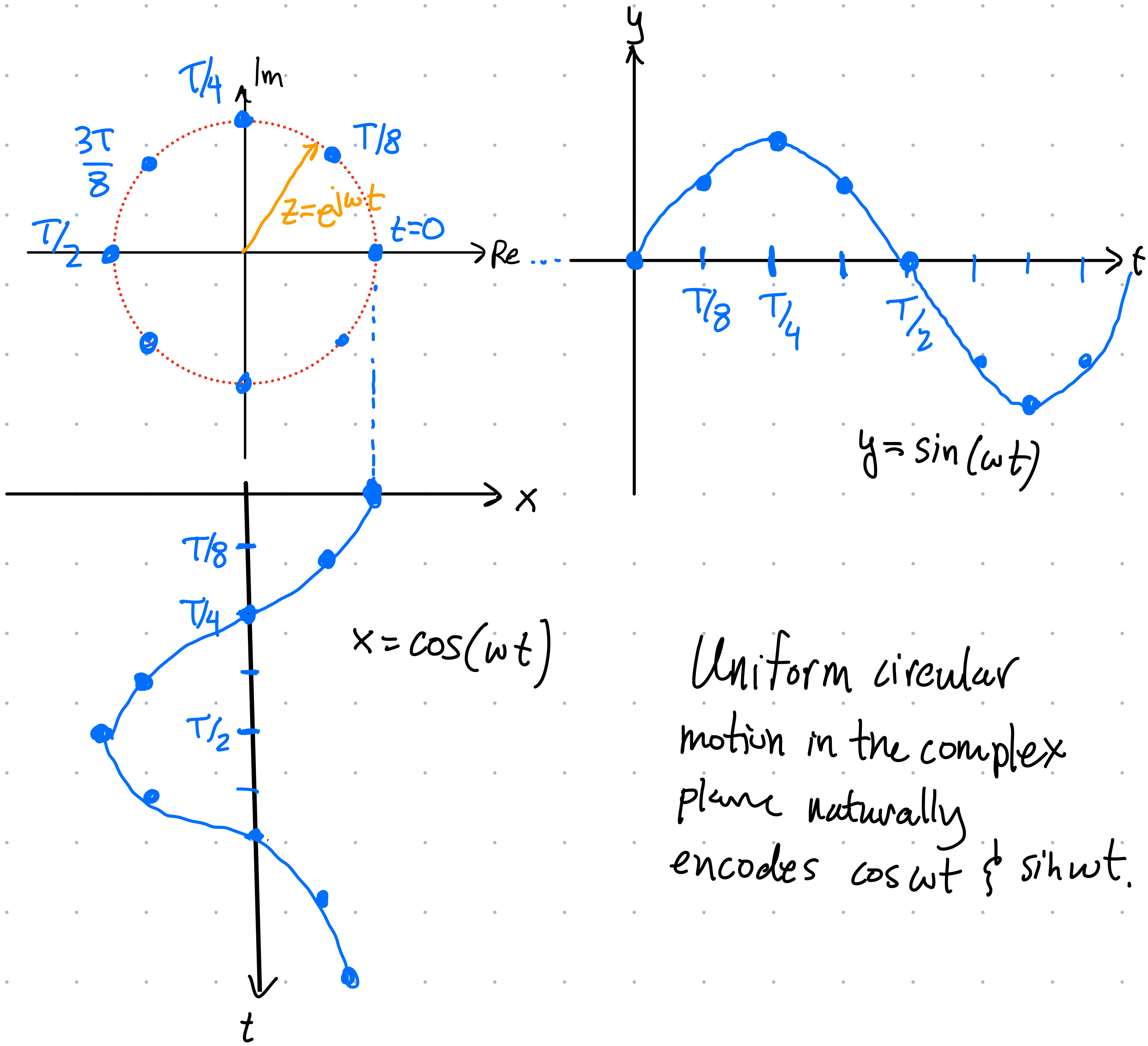
$\rightarrow$ defines a unit circle in the complex.




Suppose we allow $\phi$ to increase linearly w/ time:

$$\phi = \omega t = \frac{2\pi}{T} t = 2\pi\left(\frac{t}{T}\right)$$

As $t$ grows so to does $\phi \to$ uniform circular motion.




Uniform circular motion in the complex plane naturally encodes $\cos\omega t$ & $\sin\omega t$.

We've seen that uniform circular motion is associated w/ SHM. Can we go the other way? Can we start from SHM (mass on a spring) and connect to a circle?


$$E = \underbrace{\frac{1}{2} m v^2}_{\text{K.E.}} + \underbrace{\frac{1}{2} k x^2}_{\text{P.E.}}$$

Multiply by $\frac{2}{k}$

$$\frac{2E}{k} = \underbrace{\frac{m}{k}}_{\frac{1}{\omega^2}} v^2 + x^2$$

Recall the eq'n of motion $F = ma$

$$\rightarrow -kx = m\frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2x}{dt^2} = -\underbrace{\frac{k}{m}}_{\omega^2} x$$

$$\therefore \frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{2E}{k} = \left(\frac{v}{\omega}\right)^2 + x^2$$

$$\therefore \quad \underbrace{x^2 + \left(\frac{v}{\omega}\right)^2 = \left(\sqrt{\frac{2E}{k}}\right)^2}_{} \rightarrow \text{like } x^2+y^2=R^2$$

circle of radius $\sqrt{\frac{2E}{k}}$

on a plot of $\frac{v}{\omega}$ vs $x$