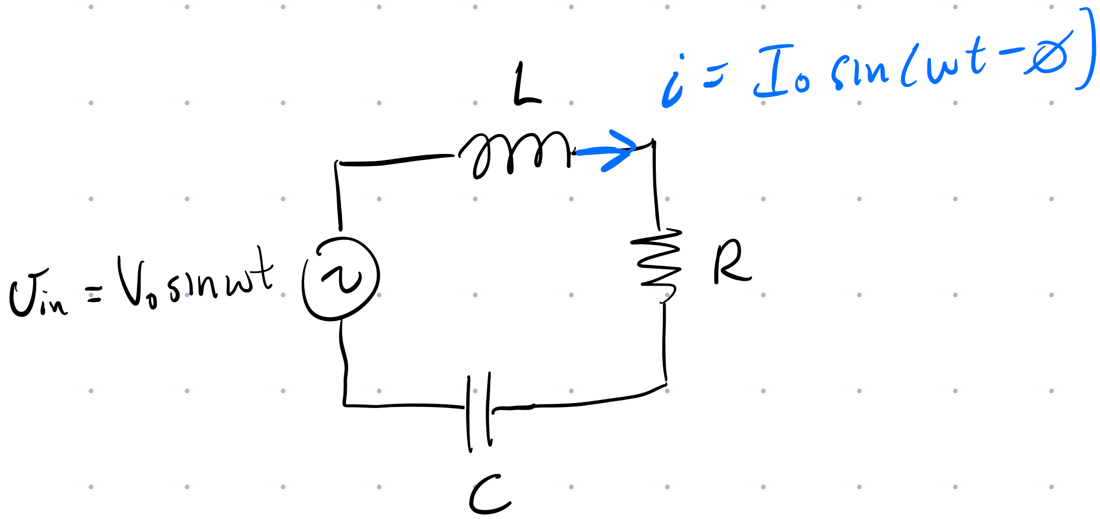
PHYS 231

Oct. 16, 2024

Last Time: LRC circuit driven by sinusoidal input



Goal: Find I_0 & ϕ of current.

1. Write $v_{in} = V_0 e^{j\omega t}$
   $i = I_0 e^{j(\omega t - \phi)}$

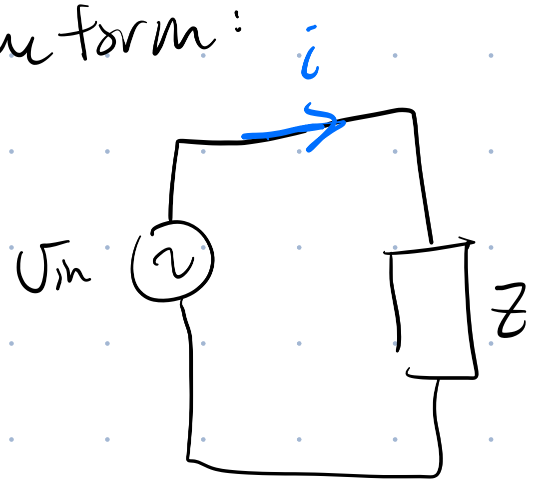
   where $e^{jx} = \cos x + j \sin x$

2. Define impedance of L, R, & C.

   $Z_L = j\omega L$, $Z_R = R$, $Z_C = \dfrac{1}{j\omega C}$

Impedances combine in series & parallel in the same way as resistors.

For *any* circuit that can be drawn in the form:



(any comb. of L, R, & C's)

$$\boxed{I_0 = \frac{V_0}{|Z|}} \qquad \tan\phi = \frac{\text{Im}[Z]}{\text{Re}[Z]}$$

For series LRC circuit

$$Z = R + j\omega L + \frac{j}{j}\,\frac{1}{j\omega C} = \underbrace{R}_{x=\text{Re}[Z]} + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{y=\text{Im}[Z]}$$

$$Z = x + jy = |Z|e^{j\phi}$$

$$|Z| = \sqrt{x^2+y^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = R\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}$$

$$\boxed{\therefore\; I_0 = \frac{V_0}{|Z|} = \frac{V_0/R}{\sqrt{1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2}}}$$

The current amp. & phase in LRC circuit.

$$\boxed{\tan\phi = \frac{y}{x} = \frac{\omega L}{R} - \frac{1}{\omega RC}}$$

$$v_R = iR \Rightarrow V_R = I_0 R$$

resistor volt. amp.

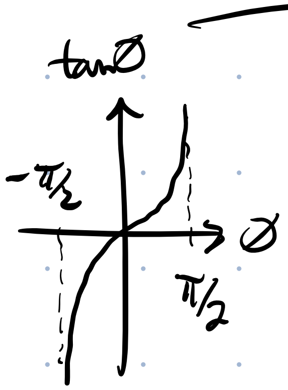
signal gen. amp

$$\boxed{\frac{V_R}{V_0} = \frac{I_0 R}{V_0} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2}}}$$

You will meas. this in Lab 4(b).

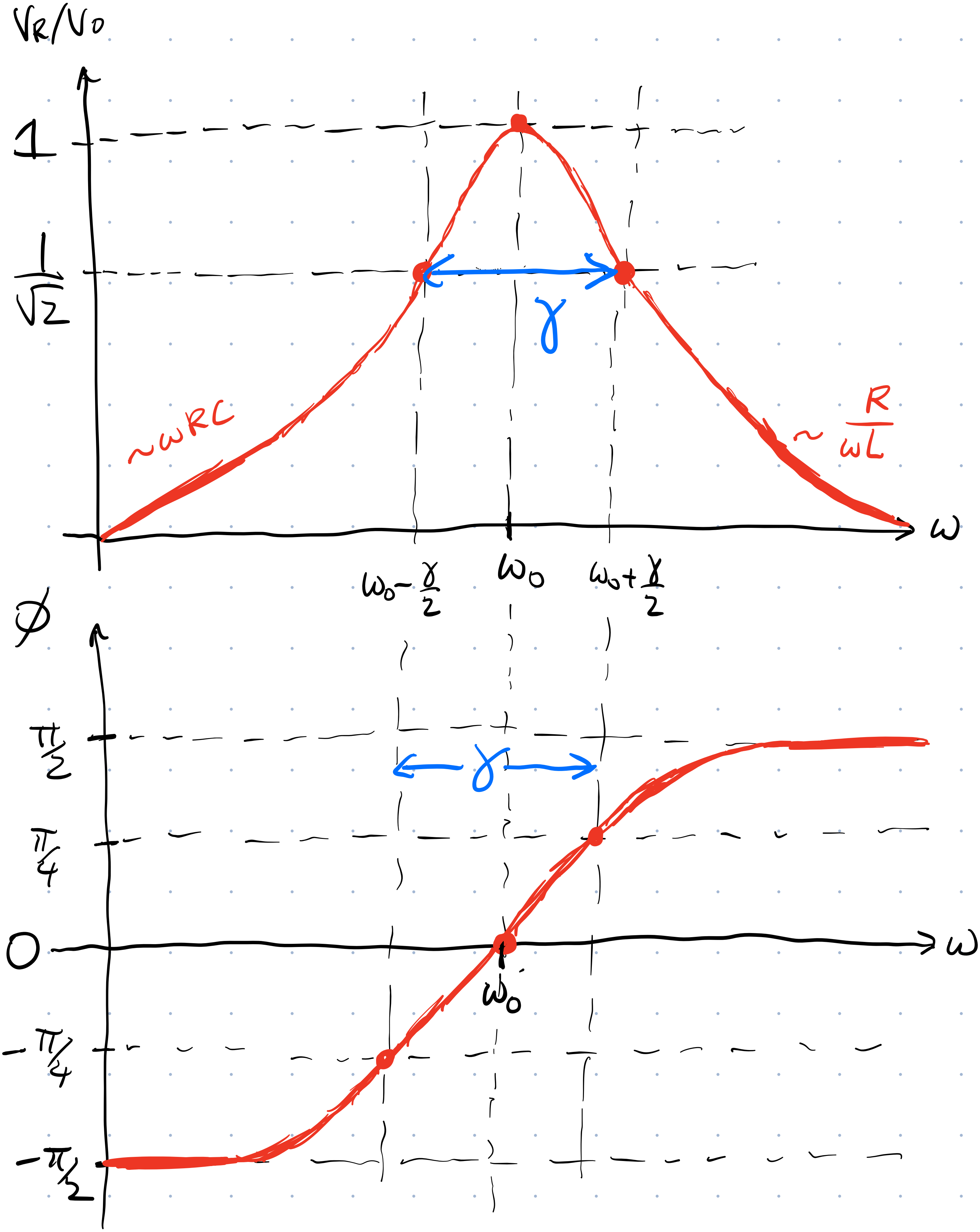
Examine freq. dependence of $\frac{V_R}{V_0}$ & ϕ.

<u>$\omega \to 0$ case</u>



$$\tan\phi = \frac{\omega L}{R} - \frac{1}{\omega RC} \approx -\frac{1}{\omega RC}$$

$$\Rightarrow \phi \to -\frac{\pi}{2}$$



V_R/V_0
1
$\frac{1}{\sqrt{2}}$
γ
$\sim \omega RC$
$\sim \frac{R}{\omega L}$
ω
$\omega_0 - \frac{\gamma}{2}$
ω_0
$\omega_0 + \frac{\gamma}{2}$
ϕ
$\pi/2$
$\pi/4$
0
$-\pi/4$
$-\pi/2$
γ
ω
ω_0

$\omega \to 0$ case

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \approx \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}$$

$$\approx \frac{1}{\frac{1}{\omega RC}} = \omega RC \quad \text{(linear)}$$

$\omega \to \infty$ case

$$\tan\phi = \frac{\omega L}{R} - \frac{1}{\omega RC} \approx \frac{\omega L}{R}$$

$$\Rightarrow \phi = +\pi/2$$

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \approx \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \approx \frac{R}{\omega L}$$

Resonance condition

Start w/ $$\frac{V_R}{V_0} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}}$$

The ratio $\frac{V_R}{V_0}$ is a maximum when $\left(\frac{\omega_0 L}{R} - \frac{1}{\omega_0 RC}\right) = 0$

The maximum occurs when $\omega = \omega_0$, where $\omega_0 = 2\pi f_0$ is the resonance freq.

$$\boxed{\omega_0 \frac{L}{\cancel{R}} - \frac{1}{\omega_0 \cancel{R} C}} = 0$$

solve for ω_0:

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

At $\omega = \omega_0$

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{1+(0)^2}} = 1$$

$$\tan\phi = \left(\boxed{\frac{\omega_0 L}{R} - \frac{1}{\omega_0 RC}}\right) = 0 \Rightarrow \phi = 0$$

when $\omega = \omega_0$

Last case: $\omega = \omega_0 \pm \frac{\gamma}{2}$ $\omega_0 = \frac{1}{\sqrt{LC}}$

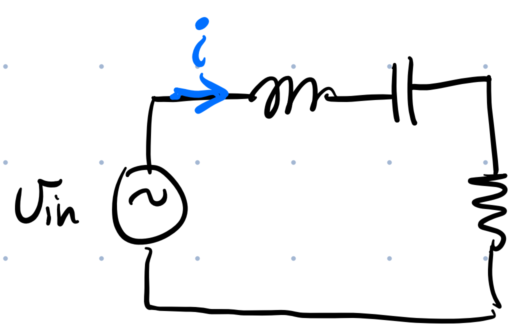
when $\omega_0 \gg \gamma$ (underdamped case) $\gamma = \frac{R}{L}$

Challenge for the student.

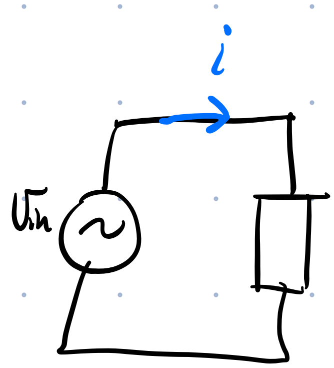
Show that when $\omega = \omega_0 \pm \frac{\gamma}{2}$

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{2}} \qquad \tan\phi = \pm 1 \Rightarrow \phi = \pm\frac{\pi}{4}$$

- Why does $\frac{V_R}{V_0} \sim \omega RC$ at low freq? ✓
- Why does $\frac{V_R}{V_0} \sim \frac{R}{\omega L}$ at high freq? ✓
- Why is $\phi = 0$ when $\omega = \omega_0$?



$\Rightarrow$



$Z = Z_R + Z_L + Z_C$

$$i = \frac{v_{in}}{Z}$$

$$v_R = iR = v_{in} \frac{R}{Z}$$

$$\therefore \frac{v_R}{v_{in}} = \frac{R}{Z} = \frac{R}{Z_R + Z_L + Z_C} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

$\omega \to 0$ case. In this case $\frac{1}{j\omega C}$ dominates (large)

$$\therefore \frac{v_R}{v_{in}} \approx \frac{R}{\frac{1}{j\omega C}} = j\omega RC$$

Look at Euler's eq'n:

$$e^{j\phi} = \cos\phi + j\sin\phi$$

consider $\phi = \pi$

$$e^{j\pi} = \underbrace{\cos\pi}_{-1} + \cancelto{0}{j\sin\pi} \Rightarrow e^{j\pi} = -1$$

consider $\varnothing = \frac{\pi}{2}$

$$e^{j\frac{\pi}{2}} = \cancel{\cos\frac{\pi}{2}}^{\,0} + j\underbrace{\sin\pi/2}_{1}$$

$$\boxed{e^{j\frac{\pi}{2}} = j}$$

$$\therefore \frac{V_R}{V_{in}} = j\omega RC \quad \text{at low freq.}$$

$$\therefore \frac{V_R}{V_{in}} = \omega RC\, e^{j\pi/2}$$

$$\frac{iR}{V_{in}} = \omega RC\, e^{j\pi/2}$$

$$\frac{I_o R\, \cancel{e^{j\omega t}}\, e^{-j\varnothing}}{V_o\, \cancel{e^{j\omega t}}} = \omega RC\, e^{j\pi/2}$$

$$\boxed{\frac{V_R}{V_o}}\, e^{\boxed{-j\varnothing}} = \boxed{\omega RC}\, e^{\boxed{j\pi/2}}$$

High Freq case $\omega \to \infty$

$$\frac{V_R}{V_{in}} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} \simeq \frac{R}{j\omega L} = \frac{R}{\omega L\, e^{j\pi/2}}$$

$$= \frac{R}{\omega L} e^{-j\pi/2}$$

$$\boxed{\frac{V_R}{V_0} = \frac{R}{\omega L} \qquad \phi = \pi/2}$$ ω large.

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when $\omega = \omega_0$

$$j\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0$$

resonance condition.

$$\frac{V_R}{V_{in}} = \frac{R}{R + \underbrace{j\omega L + \frac{1}{j\omega C}}_{=0 \text{ when } \omega = \omega_0}} = \frac{R}{R} = 1\left(e^{j0}\right)$$

$$\frac{V_R}{V_0} e^{-j\phi} = 1 e^{j0} \Rightarrow \frac{V_R}{V_0} = 1 \quad \phi = 0$$