PHYS 231 Oct. 9, 2024

Last Time:

Can represent any complex number is two ways:

$$z = x + jy = |z| e^{j\phi}$$

$x = Re[z]$ "real part of z"

$y = Im[z]$ "imaginary part of z"

Euler's Eq'n: $e^{j\phi} = \cos\phi + j\sin\phi$

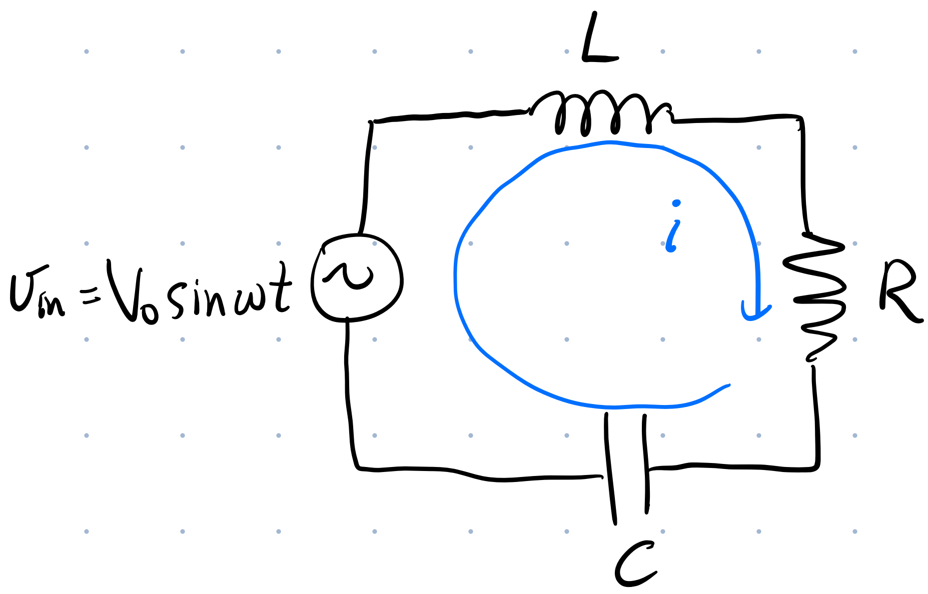
$$x = |z|\cos\phi$$
$$y = |z|\sin\phi$$

$$|z| = \sqrt{x^2+y^2}$$

$$\tan\phi = \frac{y}{x}$$

Today: Use complex nos. as a mathematical tool to help us analyze circuits driven by sinusoidal signals.

Series LRC circuit (Lab #4)



Assume that current is of the form

$$i = I_0 \sin(\omega t - \phi)$$

Find I_0 & ϕ.

KLR: $$v_{in} - v_L - v_R - v_C = 0$$

$$\therefore v_{in} = L\frac{di}{dt} + Ri + \frac{1}{C}q$$

take time derivative.

$$\boxed{\frac{dv_{in}}{dt} = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i}$$

Represent in input voltage $v_{in} = V_0 \sin \omega t$

as $$v_{in} = V_0 e^{j\omega t} = V_0\left[\cos(\omega t) + j\sin(\omega t)\right]$$

$$\mathrm{Im}\left[v_{in}\right] = V_0 \sin \omega t$$

We will do calculations using complex exponential representation of v_{in} & i. Then in the end, take imaginary part which corresponds to our original v_{in} & yields desired current.

Also represent the current $i = I_0 \sin(\omega t - \phi)$ as $i = I_0 e^{j(\omega t - \phi)}$

Need:
$$\frac{dv_{in}}{dt} = \frac{d}{dt}\left(V_0 e^{j\omega t}\right) = j\omega \underbrace{V_0 e^{j\omega t}}_{v_{in}}$$
$$= j\omega v_{in}$$

$$\frac{di}{dt} = \frac{d}{dt}\left(I_0 e^{j(\omega t - \phi)}\right) = j\omega \underbrace{I_0 e^{j(\omega t - \phi)}}_{i}$$
$$= j\omega i$$

$$\frac{d^2 i}{dt^2} = \frac{d}{dt}(j\omega i) = j\omega \frac{di}{dt} = j\omega (j\omega i)$$
$$= -\omega^2 i$$

$$\frac{dv_{in}}{dt} = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i$$

$$j\omega v_{in} = -\omega^2 L i + j\omega R i + \frac{1}{C} i$$

divide by $j\omega$

$$\therefore v_{in} = \frac{-j\omega^{\cancel{2}} L i}{j\,j\cancel{\omega}} + R i + \frac{1}{j\omega C} i$$

$$\therefore \underbrace{v_{in}}_{\text{input voltage.}} = \underbrace{j\omega L\, i}_{v_L} + \underbrace{R i}_{v_R} + \underbrace{\frac{1}{j\omega C} i}_{v_C}$$

Resistor $v_R = i \overbrace{R}^{Z_R = R}$

Inductor $v_L = i \underbrace{(j\omega L)}_{Z_L \,:\, \text{impedance of inductor.}}$

Capacitor $v_C = i \underbrace{\left(\frac{1}{j\omega C}\right)}_{Z_C \,:\, \text{impedance of capacitor.}}$

$$v_{in} = \underbrace{(Z_L + R + Z_C)}_{\text{series combination of impedances:}}\, i$$

Net series equivalent impedance.

$$Z = Z_L + R + Z_C$$

$$\boxed{i = \frac{v_{in}}{Z}}$$

write $i = I_0 e^{j(\omega t - \phi)}$

$v_{in} = V_0 e^{j\omega t}$

$$I_0 \underbrace{e^{j(\omega t - \phi)}}_{e^{j\omega t} e^{-j\phi}} = \frac{V_0 e^{j\omega t}}{Z}$$

$$I_0 \cancel{e^{j\omega t}} e^{-j\phi} = \frac{V_0 \cancel{e^{j\omega t}}}{Z}$$

$$I_0 e^{-j\phi} = \frac{V_0}{Z} \quad \#$$

Work on Z.

$$Z = j\omega L + R + \frac{j}{j}\frac{1}{j\omega C}$$

want to express Z as $|Z| e^{j\phi_Z}$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$= x + j \quad y \quad \Rightarrow$$

$x = R$ real part of imped.

$y = \omega L - \frac{1}{\omega C}$ imaginary part of imped.

$$|Z| = \sqrt{x^2 + y^2}$$

$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

factor out R^2 from $\sqrt{\quad}$?

$$\boxed{|Z| = R\sqrt{1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2}}$$

$$\tan\phi_z = \frac{y}{x} = \frac{\omega L}{R} - \frac{1}{\omega RC}$$

$$\boxed{\phi_z = \tan^{-1}\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)}$$

$$Z = x + jy = |Z|e^{j\phi_z}$$

sub $Z = |Z|e^{j\phi_z}$ into (#)

$$I_0 e^{-j\phi} = \frac{V_0}{|Z|e^{j\phi_z}} = \frac{V_0}{|Z|}e^{-j\phi_z}$$

This is valid only if $\phi = \phi_z$

$\therefore$ phase of our current in LRC circuit is :

$$\boxed{\tan\phi = \frac{\omega L}{R} - \frac{1}{\omega RC}}$$

and $$\boxed{I_0 = \frac{V_0}{|Z|} = \frac{V_0/R}{\sqrt{1+\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}}}$$

Return to (#)

$$I_0 e^{-j\phi} = \frac{V_0}{|Z|}$$ mult. by $e^{j\omega t}$

$$\underbrace{I_0 e^{j\omega t} e^{-j\phi}}_{e^{j(\omega t-\phi)}} = \frac{V_0 e^{j\omega t}}{|Z|}$$

write $$\left.\begin{aligned} e^{j\omega t} &= \cos\omega t + j\sin\omega t \\ e^{j(\omega t-\phi)} &= \cos(\omega t-\phi) + j\sin(\omega t-\phi) \end{aligned}\right\}$$ Euler's eq'n.

$$I_0\left[\cos(\omega t-\phi) + j\sin(\omega t-\phi)\right]$$
$$= \frac{V_0}{|Z|}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

We said that at the end of our manipulations, we would find our sol'n by taking the imaginary component of our result.

$$\Rightarrow \underbrace{I_0 \sin(\omega t - \phi)}_{i} = \frac{1}{|Z|} \underbrace{V_0 \sin \omega t}_{v_{in}}$$

$$\boxed{\begin{aligned} &\text{know } I_0 = \frac{V_0 / R}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \\ &\tan \phi = \frac{\omega L}{R} - \frac{1}{\omega RC} \end{aligned}}$$