PHYS 231 Oct. 7, 2024

Last Time:

define $\sqrt{-1} = \pm j$

Solution to $x^2 - 2x + 4 = 0$ is

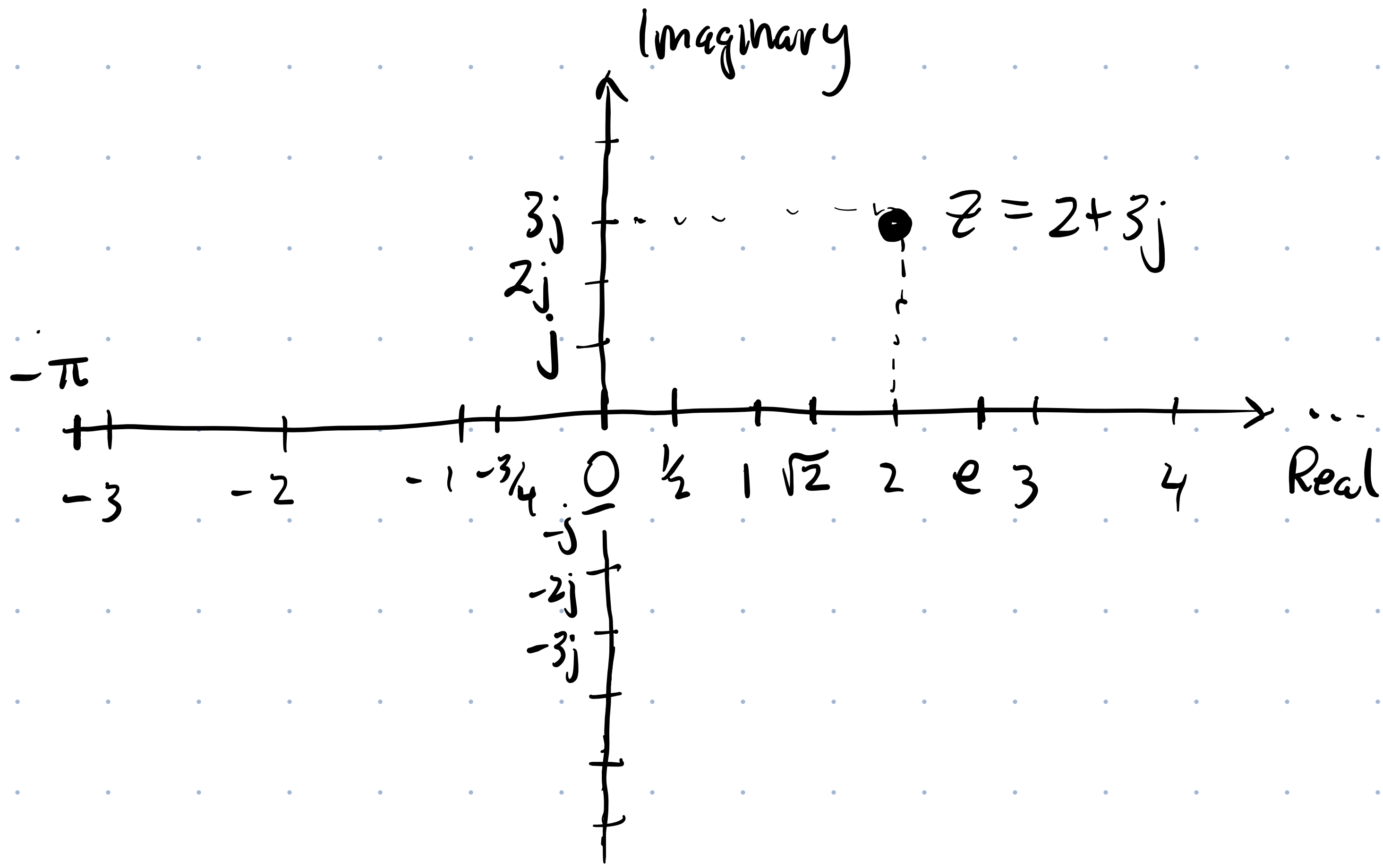
$$x = 1 \pm j\sqrt{3}$$

In general, we call the real part of a complex no. $z = x + jy$

real part of complex no. z $\quad Re[z] = x$ (anything in z that doesn't involve j)

imaginary part of z. $\quad Im[z] = y$ (anything that multiplies j, but not including j).

Draw the number line

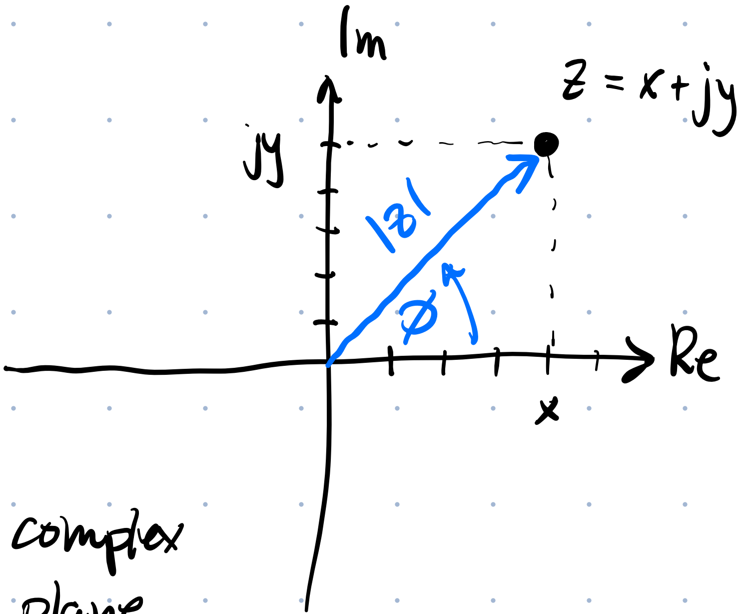


what nos. like $1 \pm j\sqrt{3}$?

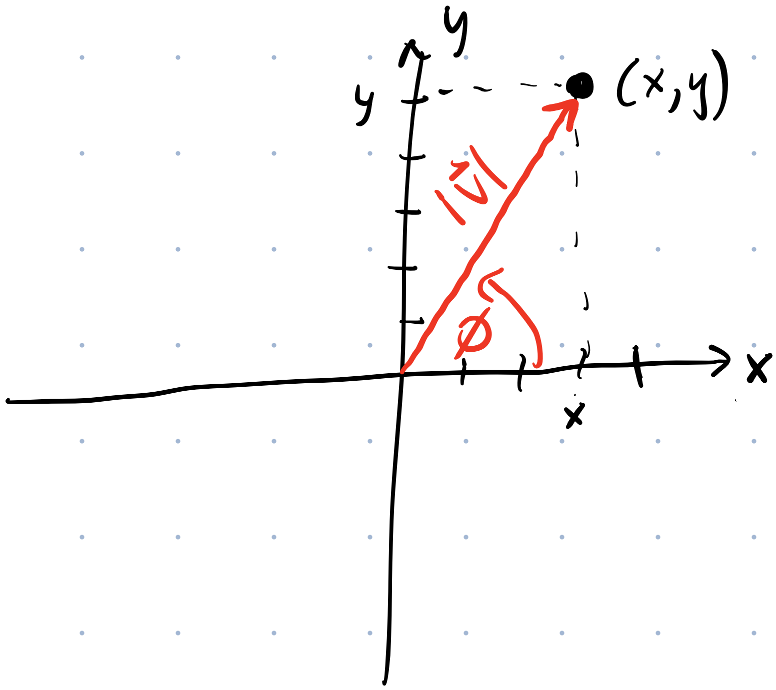
To extend this picture, create a second axis $\perp$ to the first. We'll call the origonal axis the "Real" axis & the new $\perp$ axis the imaginary axis.

Place $z = 2+3j$ on our coord. sys.

Notice that in many ways $z = x + jy$ is similar to a 2-component vector $\vec{v} = x\,\hat{i} + y\,\hat{j}$



complex plane



$$z = x + jy \quad (\#)$$

Likewise represent complex no. z as a dist. from the origin $|z|$ (magnitude of z) and an ϕ meas. ccw from real axis.

$$x = |z| \cos \phi \quad (1)$$
$$y = |z| \sin \phi \quad (2)$$

$$\vec{v} = x\,\hat{i} + y\,\hat{j}$$

$\vec{v} = |\vec{v}|$ dir'n is ϕ rotated ccw from x-axis.

$$x = |\vec{v}| \cos \phi$$
$$y = |\vec{v}| \sin \phi$$

$$|v| = \sqrt{x^2 + y^2}$$

Plug ① & ② into ④

$$\boxed{z = x + jy}$$

$$= |z|\cos\phi + j|z|\sin\phi$$

$$\boxed{z = |z|(\cos\phi + j\sin\phi)}$$

Two equil. ways of representing the same complex number.

Let's consider x^2+y^2 for our complex no. z.

From ① $x^2 = |z|^2\cos^2\phi$

② $y^2 = |z|^2\sin^2\phi$

$$\therefore x^2+y^2 = |z|^2\underbrace{(\cos^2\phi + \sin^2\phi)}_{1}$$

$$\boxed{\therefore |z| = \sqrt{x^2+y^2} = \sqrt{Re[z]^2 + Im[z]^2}}$$

Taking the ratio $\frac{y}{x} = \frac{\cancel{|z|}\sin\phi}{\cancel{|z|}\cos\phi} = \tan\phi$

$$\boxed{\tan\phi = \frac{y}{x} = \frac{\mathrm{Im}[z]}{\mathrm{Re}[z]}}$$

Summary:

Can represent any complex no. z in two ways.

(i) $z = x + jy$

(ii) $z = |z|(\cos\phi + j\sin\phi)$

If given x & y, $|z| = \sqrt{x^2+y^2}$

$\tan\phi = \frac{y}{x}$

If given $|z|$ & ϕ, $x = |z|\cos\phi$

$y = |z|\sin\phi$

Review of Taylor series (really the McLauren series)

Claim; can express any fcn that is infinitely differentiable as a power series.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Need a scheme for finding the values of the coefficients $a_0, a_1, a_2, \ldots$

Consider $f(0) = f(x)\Big|_{x=0} = a_0 + a_1(0) + a_2(0) + \ldots$

$$\boxed{\therefore\ a_0 = f(0)}$$

Next, consider $\frac{df}{dx}$

$$\frac{df}{dx} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$\boxed{\therefore\ a_1 = \frac{df}{dx}\Big|_{x=0}}$$

One more... consider $\frac{d^2f}{dx^2}$

$$\frac{d^2f}{dx^2} = 0 + 0 + 2a_2 + 3\cdot 2\, a_3 x + \dots$$

$$\boxed{\therefore\ a_2 = \frac{1}{2}\frac{d^2f}{dx^2}\bigg|_{x=0}}$$

Continuing in this fasion, we would find.

$$a_3 = \frac{1}{3\cdot 2}\frac{d^3f}{dx^3}\bigg|_{x=0} = \frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{x=0}$$

In general, $\boxed{a_n = \frac{1}{n!}\frac{d^{(n)}f}{dx^{(n)}}\bigg|_{x=0}}$

$\therefore$ Taylor series expansion of $f(x)$ about $x=0$ is then:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^{(n)}f}{dx^{(n)}}\bigg|_{x=0} x^n}$$

Example: Find Taylor series for $\sin x$.

n	a_n	$\frac{d^{(n)}}{dx^{(n)}} \sin(x)$	$\frac{1}{n!} \frac{d^{(n)}}{dx^{(n)}} \sin x \Big\vert_{x=0}$
0	a_0	$\sin x$	0
1	a_1	$\cos x$	1
2	a_2	$-\sin x$	0
3	a_3	$-\cos x$	$-1/3!$
4	a_4	$\sin x$	0
5	a_5	$\cos x$	$1/5!$
		$\vdots$	

All even values of n result in $a_n = 0$.

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots$$

Exercise for the student.

Show that the taylor series expansions for $\cos x$ & e^x are given by:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

To connect Taylor series to complex nos., consider $e^{j\phi}$. i.e. sub $x = j\phi$ into taylor series of e^x.

$$e^{j\phi} = 1 + (j\phi) + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \ldots$$

$$\therefore e^{j\phi} = 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + j\frac{\phi^5}{5!} + \ldots$$

n	j^n	
2	j^2	-1
3	j^3	$jj^2 = -j$
4	j^4	$jj^3 = 1$
5	j^5	$jj^4 = j$
6		

$$e^{j\phi} = \underbrace{\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \ldots\right)}_{\cos\phi} + j\underbrace{\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots\right)}_{\sin\phi}$$

Euler's Eq'n $\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$