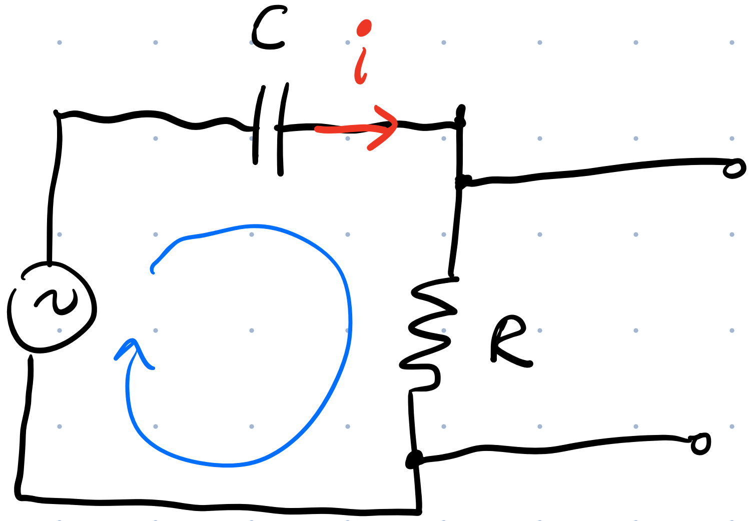
PHYS 231 Oct. 2, 2024

- No lab this week
- Assign. #2 on course website.

Last Time:

$v_{in} = V_0 \sin(\omega t)$



Loop rule:

$$v_{in} - \frac{q}{C} - iR = 0$$

take a time derivative:

$$\frac{dv_{in}}{dt} - \frac{1}{C} i - R\frac{di}{dt} = 0 \quad (\#)$$

Know $v_{in} = V_0 \sin \omega t$

assume $i = I_0 \sin(\omega t + \phi)$

$\Big\}$ sub into (#)

Want to solve for $I_0(\omega)$ & $\phi(\omega)$.

i.e. Determine how i varies w/ frequency.

start by evaluating $\frac{dv_{in}}{dt} = \omega V_0 \cos \omega t$

$\frac{di}{dt} = \omega I_0 \cos(\omega t + \phi)$

$\Big\}$ sub into (#)

$$\omega V_0 \cos \omega t = \frac{I_0}{C} \sin(\omega t + \phi) + \omega I_0 R \cos(\omega t + \phi)$$

Use the following trig identities

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

$$\cos(\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

$$\omega V_0 \underline{\cos \omega t} = \frac{I_0}{C}\left[\underline{\sin \omega t} \cos \phi + \underline{\cos \omega t} \sin \phi\right]$$

$$+ \omega I_0 R \left[\underline{\cos \omega t} \cos \phi - \underline{\sin \omega t} \sin \phi\right]$$

$$\sin \omega t \left[\underbrace{\frac{I_0}{C}\cos\phi - \omega I_0 R \sin\phi}_{2} \right]$$

$$+ \cos \omega t \left[\underbrace{\frac{I_0}{C}\sin\phi + \omega I_0 R\cos\phi - \omega V_0}_{1} \right] = 0$$

This expression must be valid at ALL times.

At some times, will have $\sin \omega t = 0$ $\cos \omega t = 1$

$\rightarrow$ in this case bracket ① $= 0$

At other times will have $\sin \omega t = 1$ $\cos \omega t = 0$

$\rightarrow$ in this case, require ② $= 0$

Start w/ ② $= 0$

$$\therefore \frac{\cancel{I_0}}{C}\cos\phi - \omega \cancel{I_0} R \sin\phi = 0$$

We've eliminated unknown I_0, solve for ϕ.

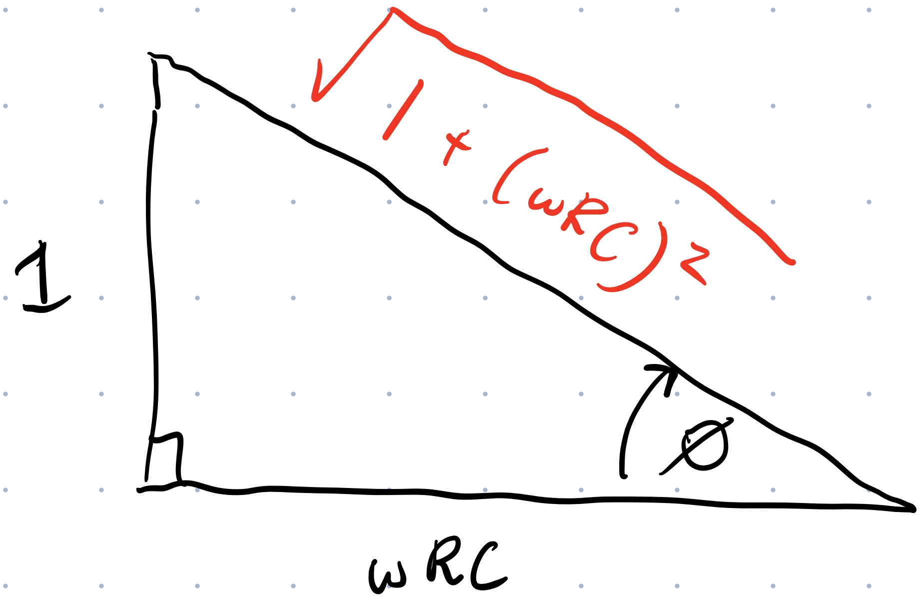
$$\frac{1}{C}\cos\phi = \omega R \sin\phi$$

$$\boxed{\therefore \tan\phi = \frac{1}{\omega RC} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{1}{\omega RC}\right)}$$

Before tackling ①, we will need to know $\cos\phi$ & $\sin\phi$.

Construct a right-angle triangle from

$$\tan\phi = \frac{opp}{adj} = \frac{1}{\omega RC}$$



With is triangle, can now find:

$$\cos\phi = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}} \qquad \sin\phi = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

Sub these expressions into ①.

$$\frac{I_0}{C}\underbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}_{\sin\phi} + \omega I_0 R \underbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}_{\cos\phi} - \omega V_0 = 0$$

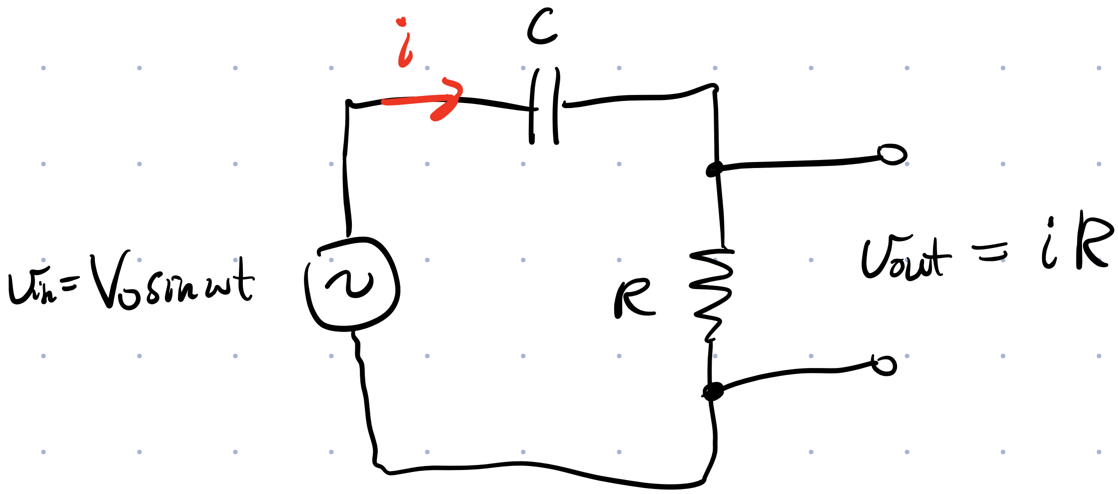
This expression only involves I_0, solve for it.

Exercise for student, show that

$$\boxed{I_0 = \frac{\omega V_0 C}{\sqrt{1+(\omega RC)^2}}} \qquad \boxed{\tan\phi = \frac{1}{\omega RC}}$$

$\therefore$ we now know the current $i = I_0 \sin(\omega t + \phi)$

Practical use of RC circuit driven by sine wave.



$$V_{out} = iR = I_0 R \sin(\omega t + \phi)$$

Consider the amplitude of V_{out} which is given by

$$I_0 R = \frac{\omega V_0 RC}{\sqrt{1+(\omega RC)^2}}$$

We could also evaluate the ratio of the output amp. $I_0 R$ divide by input amplitude V_0

$$\boxed{\frac{I_0 R}{V_0} = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}$$

consider 3 cases: (i) $\omega RC \ll 1$

(ii) $\omega RC \gg 1$

(iii) $\omega RC = 1$

(i) $\omega RC \ll 1$ $\qquad \sqrt{1+(\omega RC)^2} \approx 1$

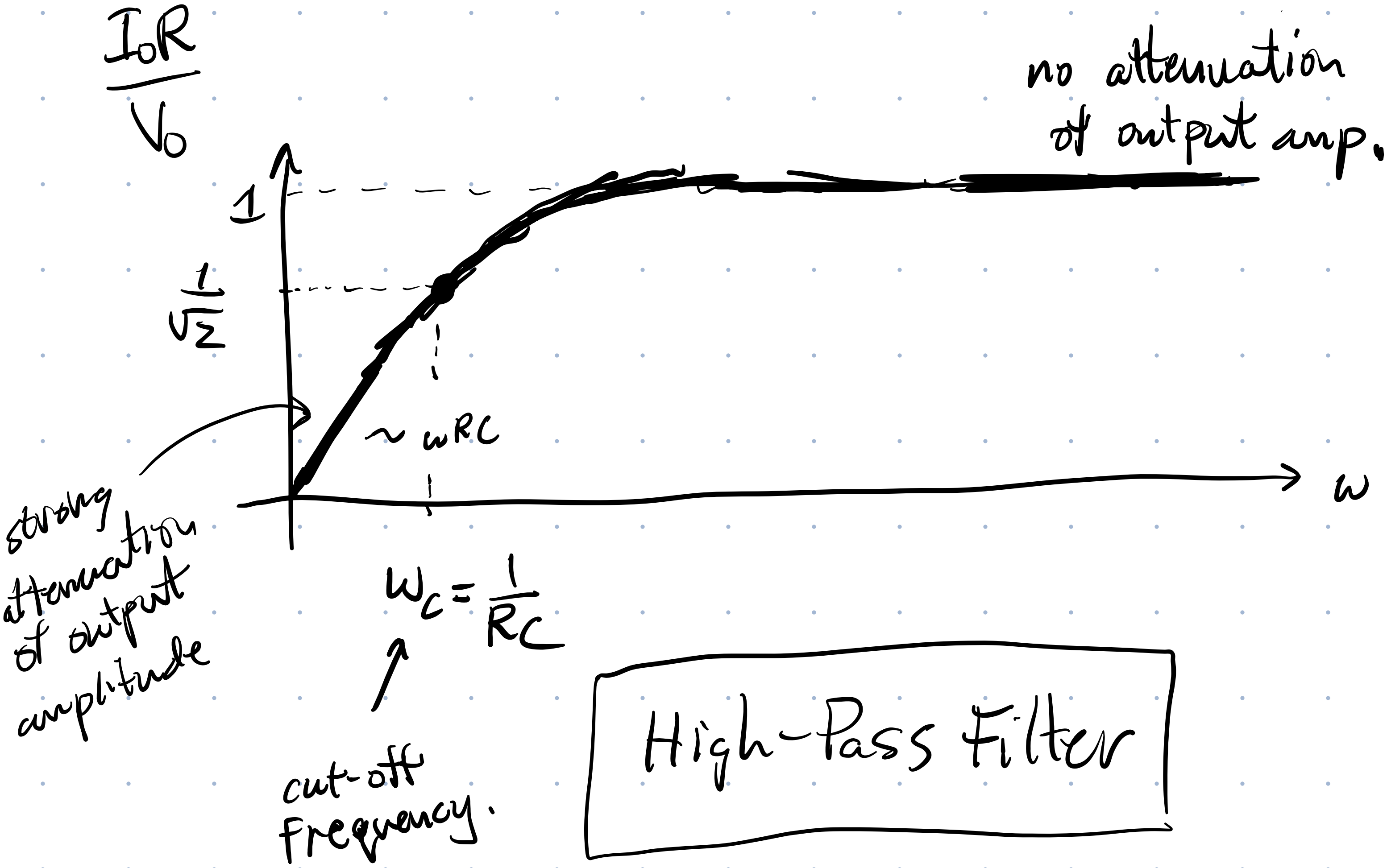
$$\frac{I_0 R}{V_0} \approx \omega RC \quad \text{(small)}$$

(ii) $\omega RC \gg 1$ $\sqrt{1+(\omega RC)^2} \approx \omega RC$

$$\frac{I_0 R}{V_0} \approx \frac{\omega RC}{\omega RC} = 1.$$

(iii) $\omega RC = 1$ $\sqrt{1+(\omega RC)^2} = \sqrt{2}$

$$\frac{I_0 R}{V_0} = \frac{1}{\sqrt{2}}$$



High-Pass Filter

Solving the RC circuit driven by a sine wave wasn't too difficult, but it involved lots of algebra, trig fcns, & a little trick to find $\cos\phi$ & $\sin\phi$.

We need new mathematical tools so that we can solve these types of problems more efficiently.

Develop the idea of complex nos.

Definition: $\sqrt{-1} = \pm j$

$$\left(\sqrt{-1}\right)^2 = -1 = (\pm j)^2 = j^2$$

$$\Rightarrow j^2 = -1 \checkmark$$

Why is this definition useful?

- it allows us to solve some problems that previously had no obvious sol'n.

Eg. consider $\sqrt{-36} = \underbrace{\sqrt{-1}}_{j}\,\underbrace{\sqrt{36}}_{6} = \pm 6j$

Eg. consider $x^2 - 2x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \begin{array}{l} a=1 \\ b=-2 \\ c=4 \end{array}$$

$$x = \frac{2 \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)} = 1 \pm \frac{\sqrt{-12}}{2}$$

$$x = 1 \pm \frac{\sqrt{-1}\sqrt{3}\,\cancel{\sqrt{4}}}{\cancel{2}}$$

$$\boxed{x = 1 \pm j\sqrt{3}}$$

check.

$$\left(x - (1+j\sqrt{3})\right)\left(x - (1-j\sqrt{3})\right)$$

$$= x^2 - x(1\cancel{-j\sqrt{3}}) - x(1\cancel{+j\sqrt{3}}) + (1+j\sqrt{3})(1-j\sqrt{3})$$

$$= x^2 - 2x + (1 \underbrace{-3j^2}_{+3}) = x^2 - 2x + 4 \quad \checkmark$$