PHYS 231 Sept. 23, 2024

Last Time: Propagation of Errors

If $y = f(x_1, x_2, \ldots, x_N)$

& meas $x_1 \pm \Delta x_1$

$x_2 \pm \Delta x_2$

$\vdots$

$x_N \pm \Delta x_N$

then

$$\Delta y = \sqrt{\left(\frac{\partial y}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial y}{\partial x_2}\Delta x_2\right)^2 + \ldots + \left(\frac{\partial y}{\partial x_N}\Delta x_N\right)^2}$$

Today: Examples of prop. of error.

1. $R = \frac{V}{I}$ meas. $V \pm \Delta V$, $I \pm \Delta I$ $\Rightarrow$ find ΔR.

First-year $\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I}$

From prop. of errors: $\Delta R = \sqrt{\left(\frac{\partial R}{\partial V}\Delta V\right)^2 + \left(\frac{\partial R}{\partial I}\Delta I\right)^2}$

$$\frac{\partial R}{\partial V} = \frac{\partial}{\partial V}\left(\frac{V}{I}\right) = \frac{1}{I}$$

$$\frac{\partial R}{\partial I} = \frac{\partial}{\partial I}\left(\frac{V}{I}\right) = -\frac{V}{I^2}$$

$$\boxed{\therefore \Delta R = \sqrt{\left(\frac{1}{I}\Delta V\right)^2 + \left(\frac{V}{I^2}\Delta I\right)^2}}$$

To compare to first-year rule, find $\frac{\Delta R}{R}$

$$\frac{\Delta R}{R} = \frac{1}{R}\sqrt{\left(\underbrace{\frac{V}{I}}_{R}\frac{\Delta V}{V}\right)^2 + \left(\underbrace{\frac{V}{I}}_{R}\frac{\Delta I}{I}\right)^2}$$

$$= \cancel{\frac{1}{R}}\sqrt{\cancel{R}^2\left(\frac{\Delta V}{V}\right)^2 + \cancel{R}^2\left(\frac{\Delta I}{I}\right)^2}$$

$$\boxed{\therefore \frac{\Delta R}{R} = \sqrt{\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2}}$$

Like the first-year rule w/ a quadrature sum.

Example $y = Ax^n$ — const

Powers. meas. $x \pm \Delta x$

Assume A, n const. w/ $\Delta A = \Delta n = 0$

First year $\dfrac{\Delta y}{y} = \left| n \dfrac{\Delta x}{x} \right|$

Prop. of errors

$$\Delta y = \sqrt{\left(\frac{\partial y}{\partial x} \Delta x \right)^2}$$

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(A x^n \right) = A n x^{n-1}$$

$$\Delta y = \sqrt{\left(A n x^{n-1} \Delta x \right)^2} = \boxed{\left| A n x^{n-1} \Delta x \right|}$$

Consider $\dfrac{\Delta y}{y} = \left| \dfrac{\cancel{A}\cancel{n} \, x^{n-1} \, \Delta x}{\cancel{A} \, x^n} \right| = \left| n \dfrac{\Delta x}{x} \right|$

$\frac{1}{x}$

first-year result.

Another example. $y = x \ln x$ meas. $x \pm \Delta x$
find Δy.

No first-year rule, but prop. of errors still works!

$$\Delta y = \sqrt{\left(\frac{\partial y}{\partial x}\Delta x\right)^2} = \left|\frac{\partial y}{\partial x}\Delta x\right|$$

$$\frac{dy}{dx} = \frac{d}{dx}(x \ln x) = \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \ln x + 1$$

$$\boxed{\therefore \Delta y = \left|(\ln x + 1)\Delta x\right|}$$

Try to justify/derive the standard error in the mean.

$$\sigma_\mu = \frac{\sigma}{\sqrt{N}}$$

std. dev.

no. of trials

std. error.

Strategy is to write down an expression for the mean μ & then apply prop. of errors.

$$\mu = \frac{1}{N}\left(x_1 + x_2 + x_3 + \ldots + x_N \right)$$

fcn of N variables.

meas. $x_1 \pm \Delta x_1$

$x_2 \pm \Delta x_2$

$\vdots$

$x_N \pm \Delta x_N$

Prop. of errors says:

$$\Delta\mu = \sqrt{\left(\frac{\partial\mu}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial\mu}{\partial x_2}\Delta x_2\right)^2 + \ldots + \left(\frac{\partial\mu}{\partial x_N}\Delta x_N\right)^2}$$

$$\frac{\partial\mu}{\partial x_1} = \frac{\partial}{\partial x_1}\left[\frac{1}{N}\left(x_1 + x_2 + \ldots + x_N\right)\right]$$

$$= \frac{1}{N}\left[\underbrace{\frac{\partial x_1}{\partial x_1}}_{1} + \cancelto{0}{\frac{\partial x_2}{\partial x_1}} + \ldots + \cancelto{0}{\frac{\partial x_N}{\partial x_1}}\right]$$

$$= \frac{1}{N}$$

Likewise, $\frac{\partial \mu}{\partial x_2} = \frac{1}{N}$, $\frac{\partial \mu}{\partial x_3} = \frac{1}{N}$ ..., $\frac{\partial \mu}{\partial x_N} = \frac{1}{N}$

$$\therefore \Delta\mu = \sqrt{\left(\frac{1}{N}\Delta x_1\right)^2 + \left(\frac{1}{N}\Delta x_2\right)^2 + \dots \left(\frac{1}{N}\Delta x_N\right)^2}$$

$$= \frac{1}{N}\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + \dots + (\Delta x_N)^2}$$

Typically each individual meas. is expected to be approx 1σ from μ (on average).

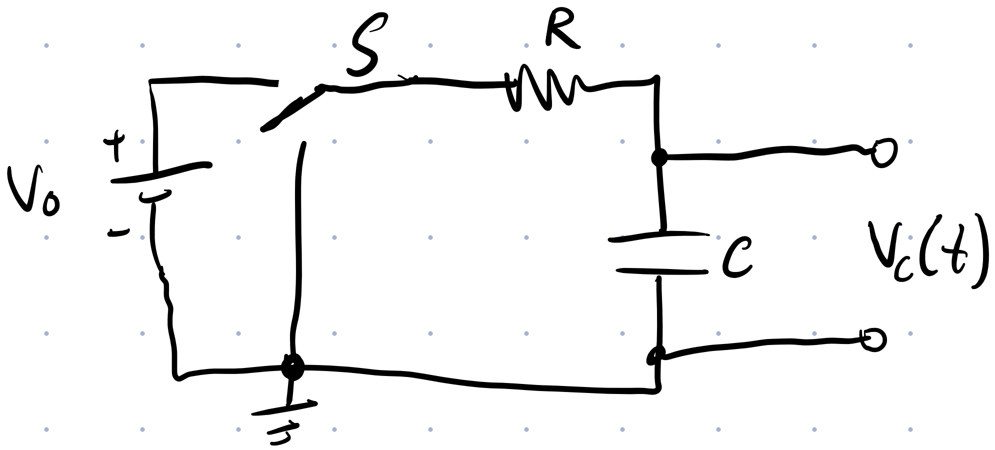
$$\therefore \Delta x_1 \approx \Delta x_2 \approx \dots = \Delta x_N \approx \sigma \quad \text{(std. dev.)}$$

$$\Delta\mu = \frac{1}{N}\sqrt{\sigma^2 + \sigma^2 + \dots + \sigma^2} = \frac{1}{N}\sqrt{N\sigma^2} = \frac{\sigma}{\sqrt{N}}$$

$$\boxed{\therefore \Delta\mu = \frac{\sigma}{\sqrt{N}}}$$ as expected!

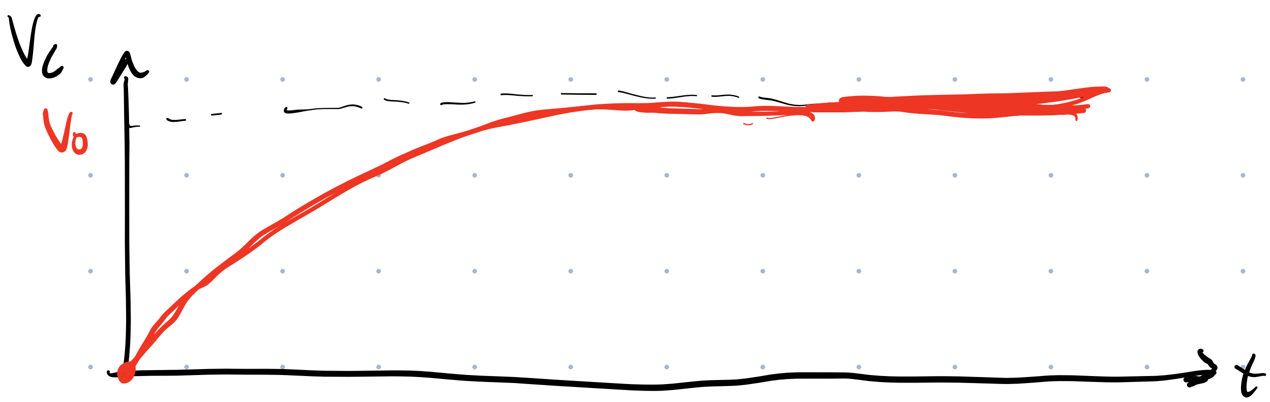
Lab #3 Preview

Charge a cap. through a resistor



If close switch at $t=0$,

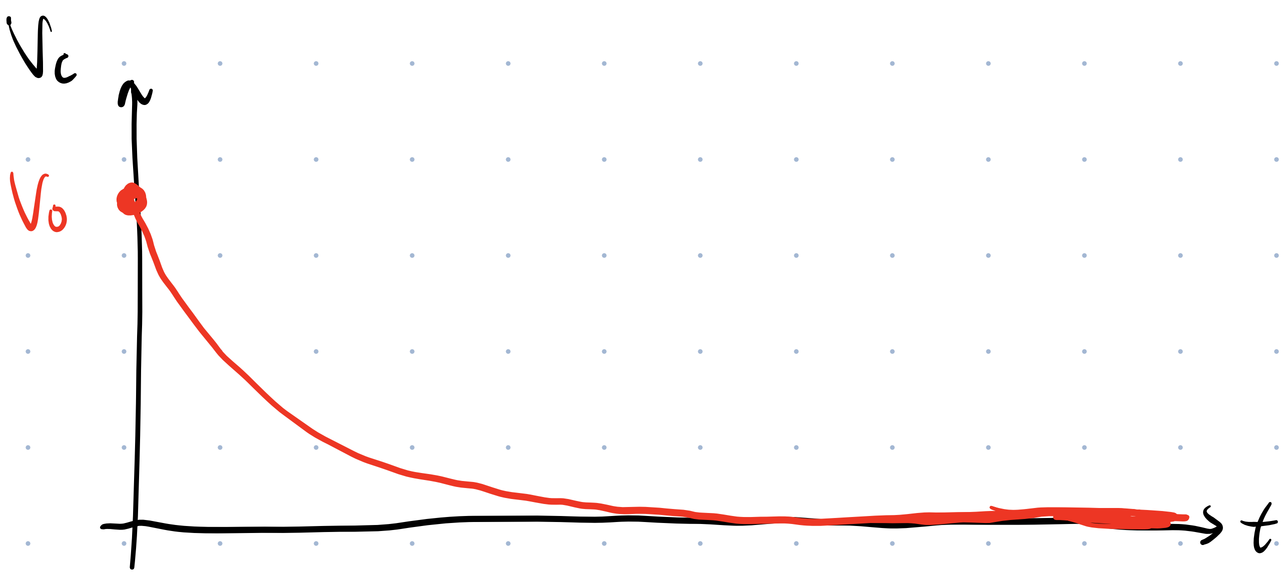
$$V_c(t) = V_0\left(1 - e^{-t/\tau}\right) \qquad \tau = RC$$



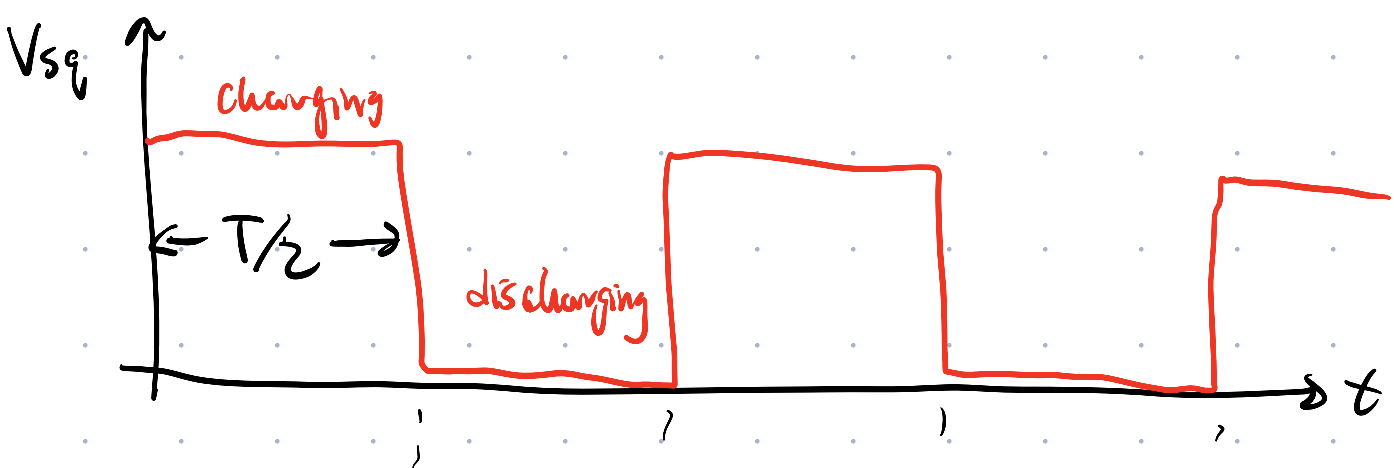
Discharging a cap. through resistor R

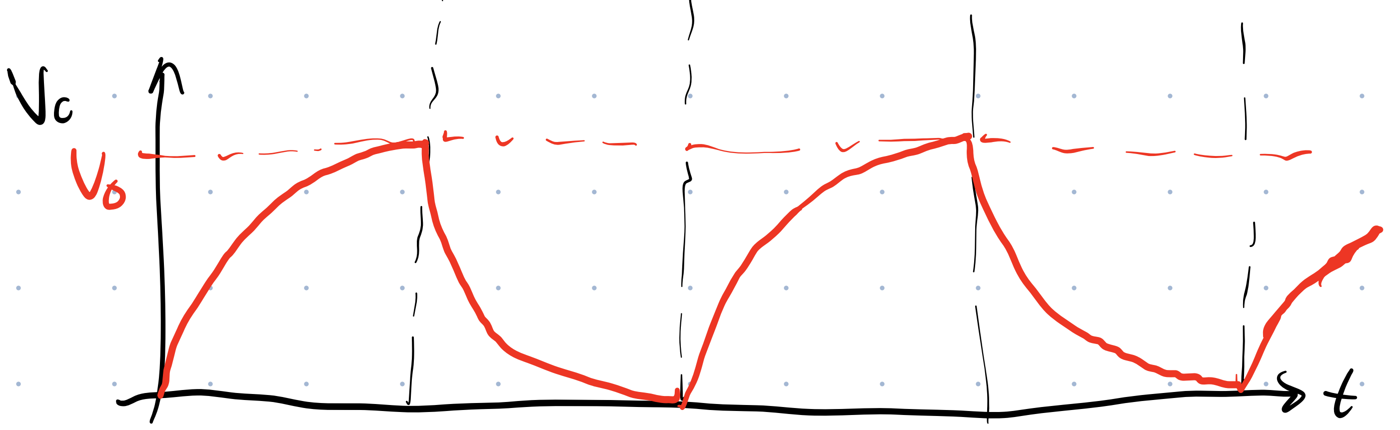
Cap. initially charged to V_0. Flip switch down @ $t=0$.

$$V_c(t) = V_0 e^{-t/\tau} \qquad \tau = RC$$



In Lab, we will use square wave to simulate a switch.





Vc
V0
t