Game Theory:
Preferences and Expected Utility

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1 Preferences

We want to examine the behavior of an individual, called a *player*, who must choose from among a set of outcomes. Begin by formalizing the set of outcomes from which this choice is to be made.

Let X be the (finite) set of outcomes with common elements x, y, z. The elements of this set are mutually exclusive (choice of one implies rejection of the others). For example, X can represent the set of candidates in an election and the player needs to chose for whom to vote. Or it can represent a set of diplomatic and military actions—bombing, land invasion, sanctions—among which a player must choose one for implementation.

The standard way to model the player is with his *preference relation*, sometimes called a *binary relation*. The relation on X represents the relative merits of any two outcomes for the player with respect to some criterion. For example, in mathematics the familiar weak inequality relation, '$\geq$', defined on the set of integers, is interpreted as "integer x is at least as big as integer y" whenever we write $x \geq y$. Similarly, a relation "is more liberal than," denoted by 'P', can be defined on the set of candidates, and interpreted as "candidate x is more liberal than candidate y" whenever we write xPy.

More generally, we shall use the following notation to denote strict and weak preferences. We shall write $x \succ y$ whenever we mean that x is strictly preferred to y and $x \succeq y$ whenever we mean that x is weakly preferred to y. We shall also write $x \sim y$ whenever we mean that the player is indifferent between x and y. Notice the following logical implications:

$$\begin{aligned} x \succ y &\Leftrightarrow x \succeq y \wedge \neg(y \succeq x) \\ x \sim y &\Leftrightarrow x \succeq y \wedge y \succeq x \\ x \succeq y &\Leftrightarrow \neg(y \succ x). \end{aligned}$$

Suppose we present the player with two alternatives and ask him to rank them according to some criterion. There are four possible answers we can get:

1. x is better than y and y is not better than x
2. y is better than x and x is not better than y
3. x is not better than y and y is not better than x
4. x is better than y and y is better than x

Although logically possible, the fourth possibility will prove quite inconvenient, so we immediately exclude it with a basic assumption.

ASSUMPTION 1. Preferences are **asymmetric**: There is no pair x and y from X such that $x \succ y$ and $y \succ x$.

We shall also require the player to be able to make judgments about every option that is of interest to us. In particular, he should be able to compare a third option, z, to the original two options. This assumption is quite strong for it implies that the player cannot refuse to rank an alternative.

ASSUMPTION 2. Preferences are **negatively transitive**: If $x \succ y$, then for any third element z, either $x \succ z$, or $z \succ y$, or both.

To understand what this assumption means, observe that it requires the player to rank z with respect to both x and y. The easiest way to illustrate this is by placing the alternatives along a line such that $x \succ y$ implies x is to the right of y. Then, we have the three possible rankings of z from the assumption: (i) $(x \succ z) \wedge \neg(z \succ y)$, (ii) $(z \succ y) \wedge \neg(x \succ z)$; and (iii) $(x \succ z) \wedge (z \succ y)$. Each is shown in the picture in Figure 1.

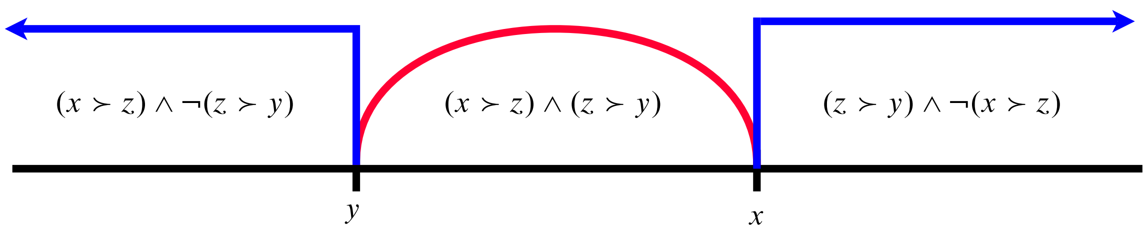


Figure 1: Illustration of Negative Transitivity.

As you can see in , this covers all possibilities of placing z somewhere along that line. For example, if (i) is true, then z is the left of y, if (ii) is true, then z is to the right of x, and if (iii) is true, z is between y and x.

There are several properties that we now define and all of which are implied by the two assumptions above. The three properties are:

1. *Irreflexivity:* For no x is $x \succ x$.

2. *Transitivity:* If $x \succ y$ and $y \succ z$, then $x \succ z$.

3. *Acyclicity:* If, for a given finite integer n, $x_1 \succ x_2, x_2 \succ x_3, \ldots, x_{n-1} \succ x_n$, then $x_n \neq x_1$.

Let's prove that transitivity is implied by negative transitivity and asymmetry. Suppose that the two properties are satisfied, and assume (a) $x \succ y$, and (b) $y \succ z$. Prove $x \succ z$.

1.	$\neg[z \succ y]$	from (b), asymmetry
2.	$[x \succ z] \vee [z \succ y] \vee [[x \succ z] \wedge [z \succ y]]$	from (a), negative transitivity
3.	$[x \succ z] \vee [[x \succ z] \wedge [z \succ y]]$	from (1) and (2), disjunctive syllogism
4.	$[[x \succ z] \vee [x \succ z]] \wedge [[x \succ z] \vee [z \succ y]]$	from (3), distribution
5.	$[x \succ z] \vee [x \succ z]$	from (4), simplification
6.	$x \succ z$	from (5), tautology

By the rule of conditional proof, we're done. Hence, asymmetry and negative transitivity imply transitivity. The following proposition states some other implications of these two assumptions for the weak and indifference preference relations.

PROPOSITION 1. *If '$\succ$' is asymmetric and negatively transitive, then*

1. $\succeq$ *is* ***complete****: For all* $x, y \in X, x \neq y$*, either* $x \succeq y$ *or* $y \succeq x$ *or both;*

2. $\succeq$ *is* ***transitive****: If* $x \succeq y$ *and* $y \succeq z$*, then* $x \succeq z$*;*

3. $\sim$ *is* ***reflexive****: For all* $x \in X$*,* $x \sim x$

4. $\sim$ *is* ***symmetric****: For all* $x, y \in X$*,* $x \sim y$ *implies* $y \sim x$*;*

5. $\sim$ *is* ***transitive****: If* $x \sim y$ *and* $y \sim z$*, then* $x \sim z$*;*

6. *If* $w \sim x, x \succ y$*, and* $y \sim z$*, then* $w \succ y$ *and* $x \succ z$*.* □

An important concept we hear a lot about is that of rationality. Here is its precise definition in terms of preference relations:

DEFINITION 1. The preference relation $\succeq$ is **rational** if it is complete and transitive.

This means that given *any* two alternatives, the individual can determine whether he likes one at least as much as the other (completeness) and no sequence of pairwise choices will result in a cycle (transitivity). As the above proposition shows, these two properties are implied by the more basic asymmetry and negative transitivity of the strict preference relation. This is the precise definition of rationality we shall use in this course.

If you are interested in social choice theory, the next step is to examine conditions that preference relations must meet in order for the set X to have maximal elements. However, for our purposes, we skip right to utility representations instead.

2 Utility Representation

Consider a set of alternatives X. A utility function $u(x)$ assigns a numerical value to $x \in X$, such that the rank ordering of these alternatives is preserved. More formally,

DEFINITION 2. A function $u : X \to \mathbb{R}$ is a **utility function representing preference relation** $\succeq$ if the following holds for all $x, y \in X$:

$$x \succeq y \Leftrightarrow u(x) \geq u(y).$$

There are many utility functions that can represent the same preference relation. Note that preferences are *ordinal*, that is, they specify the ranking of the alternatives, but not how far apart they are from each other (intensity). *Cardinal* properties are those that are only preserved under strictly increasing transformations. For example, because the function assigns numerical values to various alternatives, the magnitude of any differences in the utility between two alternatives is cardinal.

We want to use the utility function because instead of examining conditions under which preference relations produce maximal elements for a set of alternatives, it is easier to specify the numerical representation and then apply standard optimization techniques to find the maximum. Thus, the "best" options from the set X are precisely the options that have the maximum utility.

At this point, it is worth emphasizing that *players do not have utility functions*. Rather, they have preferences, which we can represent (for analytical purposes) with utility functions.

We now want to know when a given set of preferences admits a numerical representation. Not surprisingly, the result is closely linked to rationality.

PROPOSITION 2. *A preference relation $\succeq$ can be represented by a utility function only if it is rational.* □

Proof. To prove this proposition, we must show that rationality is a necessary condition for representability. In other words, if there exists a utility function that represents preferences $\succeq$, then $\succeq$ must be complete and transitive. So, assume that $u(\cdot)$ represents $\succeq$.

Completeness. Because $u(\cdot)$ is a real-valued function defined on X, for any $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. Because $u(\cdot)$ is a utility function representing the relation $\succeq$, this implies that either $x \succeq y$ or $y \succeq x$. Hence, $\succeq$ must be complete.

Transitivity. Suppose $x \succeq y$ and $y \succeq z$. Because $u(\cdot)$ represents $\succeq$, we have $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Therefore, $u(x) \geq u(z)$, which in turn implies $x \succeq z$. ■

One may wonder if any rational preference ordering $\succeq$ can be represented by some utility function. In general, the answer is no. However, if X is finite, then we can always represent a rational preference ordering with a utility function. The following proposition summarizes these results for the more basic primitives.

PROPOSITION 3. *If the set X on which $\succ$ is defined is finite, then $\succ$ admits a numerical representation if, and only if, it is asymmetric and negatively transitive.* □

Proof (Sketch of Proof). We have to show *necessity*: $\succ$ admits a numerical representation only if it is asymmetric and negatively transitive; and *sufficiency*: if $\succ$ is asymmetric and negatively transitive and X is finite, then $\succ$ admits a numerical representation.

We proved necessity for $\succeq$, so this step is analogous. Note that it does not require restricting X to be finite. To prove sufficiency, consider the preferences over any two alternatives, and generalize from there (which you can do because X is finite). ■

Thus, we know that given a rational preference ordering over a finite set of alternatives, we can always find a utility function that can represent this ordering. Alternatively, if a utility function represents a preference ordering, then that ordering must be rational.

3 Choice Under Uncertainty

Until now, we have been thinking about preferences over alternatives. That is, choices that result in certain outcomes. However, most interesting applications deal with occasions when the player may be uncertain about the consequences of choices at the time the decision is made. For example, when you choose to buy a car, you are not sure about its quality. When you chose to start a fight, you may be uncertain about whether you will win or lose.

3.1 Lotteries

Imagine that a decision-maker faces a choice among a number of risky alternatives. Each alternative may result in a number of possible **outcomes**, but which of these outcomes will occur is uncertain at the time the choice is made.

The von Neumann-Morgenstern (vNM) Expected Utility Theory models uncertain prospects as probability distributions over outcomes. These probabilities are given as part of the description of the outcomes. Thus, X is now a set of outcomes but there is also a larger set of *probability distributions* over these outcomes denoted by $\mathscr{P}$. We shall mostly be dealing with *simple* probability distributions (these involve only a finite number of possible outcomes).

DEFINITION 3. A **simple probability distribution** p on X is specified by:

1. a finite subset of X, called the **support** of p and denoted by $\text{supp}(p)$; and

2. for each $x \in \text{supp}(p)$, a number $p(x) > 0$, with $\sum_{x \in \text{supp}(p)} p(x) = 1$.

The set of simple probability distributions on X will be denoted by $\mathscr{P}$.

We shall call p (the probability distributions) also *lotteries*, and *gambles* interchangeably. In perhaps simpler words, X is the set of outcomes and p is a set of probabilities associated with each possible outcome. All of these probabilities must be nonnegative and they all must sum to 1. P then is the set of all such lotteries.

For example, suppose I am participating in a game where I could either roll a die or flip a coin. If I roll the die and the number that comes up is less than 3, I get \$120, otherwise, I get nothing. If I flip the coin and it comes up heads, I get \$100, and if it comes up tails, I get nothing. In our lingo, the set of outcomes is $X = \{0, 100, 120\}$.

The roll of the die is one objective probability distribution (lottery), which assigns the outcome \$0 a probability $2/3$, and the outcome \$120 a probability $1/3$. The support of this lottery consists of these two outcomes only. That is, the outcome \$100 is not in the support, which is another way of saying that this lottery assigns it a probability of zero. Let's denote this lottery by p. We now have $p(0) = 2/3$, $p(120) = 1/3$, and $p(100) = 0$.

The coin flip is another lottery. Let's denote it by q. The support of this lottery consists of the outcomes \$0 and \$100. Because the coin is fair, we have $q(0) = q(100) = 1/2$, and $q(120) = 0$.

In a simple lottery, the outcomes that result are certain. A straightforward generalization is to allow outcomes that are simple lotteries themselves. Suppose now we have two simple probability distributions, p and q, and some number $\alpha \in [0, 1]$. These can form a new probability distribution, r, called a *compound lottery*, written as $r = \alpha p + (1 - \alpha)q$. This requires two steps:

1. $\text{supp}(r) = \text{supp}(p) \cup \text{supp}(q)$

2. for all $x \in \text{supp}(r)$, $r(x) = \alpha p(x) + (1 - \alpha)q(x)$, where $p(x) = 0$ if $x \notin \text{supp}(p)$ and $q(x) = 0$ if $x \notin \text{supp}(q)$

That is, if an outcome is in the support of either one of the simple lotteries, it is also in the support of the compound lottery. The probability associated with an outcome in the compound lottery is a linear combination of the probabilities for this outcome from the simple lotteries.

DEFINITION 4. Given K simple lotteries p_i, and probabilities $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$, the **compound lottery** $(p_1, \ldots, p_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery p_i with probability α_i for all $i = 1, \ldots, K$.

Returning to our example, we have two simple lotteries, p (the die) and q (the coin), so $K = 2$. So we need two probabilities, α_1 and α_2, such that $\alpha_1 + \alpha_2 = 1$. To put it simply, we would have $\alpha_1 = \alpha$, and $\alpha_2 = 1 - \alpha$. The probability α is the probability of choosing the die lottery. Its complement is the probability of choosing the coin lottery. Let's suppose that α is determined by the roll of two dice such that α is the probability of their sum equaling either 5 or 6. That is, $\alpha = 1/4$.[1] Thus, $(p, q; 1/4, 3/4)$ is the compound lottery where the simple lottery p occurs with probability $1/4$, and the simple lottery q occurs with probability $1 - 1/4 = 3/4$.

For any compound lottery, we can calculate a corresponding **reduced lottery**, which is a simple lottery that generates the same probability distribution over the outcomes. In other words, we can reduce any compound lottery to a simple lottery.

DEFINITION 5. Let $(p_1, \ldots, p_K; \alpha_1, \ldots, \alpha_K)$ denote some compound lottery consisting of K simple lotteries. $\hat{p}$ is the reduced lottery that generates the same probability distribution over outcomes, and it is defined as follows. For each $x \in X$,

$$\hat{p}(x) = \sum_{i=1}^{K} \alpha_i p_i(x).$$

That is, to get a probability of some outcome x in the reduced lottery, you multiply the probability that each lottery p_i arises, α_i, by the probability that p_i assigns to the outcome x, $p_i(x)$, and then adding over all i.

Returning to our example, let's calculate the reduced lottery associated with our compound lottery induced by the roll of the two dice. We have three outcomes, and therefore:

$$\hat{p}(0) = \alpha p(0) + (1-\alpha)q(0) = (1/4)(2/3) + (3/4)(1/2) = 13/24$$
$$\hat{p}(100) = \alpha p(100) + (1-\alpha)q(100) = (1/4)(0) + (3/4)(1/2) = 9/24$$
$$\hat{p}(120) = \alpha p(120) + (1-\alpha)q(120) = (1/4)(1/3) + (3/4)(0) = 2/24$$

Clearly, $\sum_{x \in X} \hat{p}(x) = 1$, as required. Note further that $\text{supp}(\hat{p}) = \text{supp}(p) \cup \text{supp}(q)$. Thus, the simple lottery that assigns probability $13/24$ to getting nothing, $3/8$ to getting \$100, and $1/12$ to getting \$120, generates the same probability distribution over the outcomes as the compound lottery.

Using this principle, we can define more complicated lotteries that consist of compound lotteries themselves. It should be obvious how to extend the method to these cases.

[1] You should verify that you can calculate this probability. There are 4 ways to get a sum of 5 and 5 ways to get a sum of 6. The probability of getting either one or the other is $4/36 + 5/36 = 1/4$.

3.2 Preferences Over Lotteries

We now have a way of modeling risky alternatives. The next step is to define the preferences over them. We shall assume that for any risky alternative, only the reduced lottery over outcomes is of relevance to decision-makers. This is known as the *consequentialist premise*. It does not matter whether probabilities arise from simple, compound, or complex compound lotteries. The only thing that should matter for the decision-maker is the probability distribution over the outcomes, not how it arises.

In our example, the consequentialist hypothesis requires that I view the compound lottery $(p, q; 1/4, 3/4)$ and the reduced lottery $\hat{p} = (13/24, 9/24, 2/24)$ as equivalent.

More generally, it requires that I view *any* two lotteries that generate the same reduced lottery as equivalent. Let's go back to our example with three outcomes. Let the simple lottery $p = (p(0), p(100), p(120))$ denote the probabilities over the three outcomes. So, $p = (2/3, 0, 1/3)$ is our original roll of a single die simple lottery. Figure 2 illustrates a case where two different compound lotteries generate the same reduced lottery. The consequentialist hypothesis requires that I regard both of them as equivalent.

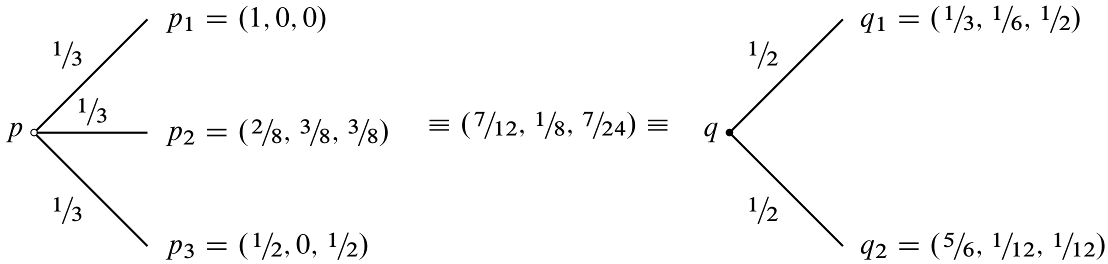


Figure 2: Two Compound Lotteries with the Same Reduced Lottery.

Before we continue with the definition of preference relations over lotteries, we note the special case of a **degenerate lottery**. This is a simple lottery that assigns probability 1 to some outcome, and 0 to all others. We denote it by p^x, where $x \in X$ is the outcome to which the lottery assigns probability 1. For example, going back to the three-outcome case, $p^0 = (1, 0, 0)$, $p^{100} = (0, 1, 0)$, and $p^{120} = (0, 0, 1)$. The degenerate lotteries will prove quite useful for replacing outcomes with equivalent lotteries. Why? Because for any outcome x we can get the appropriate degenerate lottery p^x. By the consequentialist premise, a decision maker would not care whether he is receiving outcome x directly or as a result of a lottery that yields it with certainty.

We now proceed just like we did in the case of preference relations. We take the set of alternatives, denoted (as you should recall) by $\mathcal{P}$, to be *the set of all simple lotteries over the set of outcomes* X. Next, assume that the decision maker has a preference relation $\succ$ defined on $\mathcal{P}$. (Recall that we can construct both weak and indifference relations from the strict one.) As before, we assume that this relation is rational. It is important to remember that we cannot *derive* the preferences over lotteries from preferences over outcomes, we have to *assume* them as part of the description of the model. These assumptions are represented by a set of three axioms: R (Rationality), C (Continuity), and I (Independence).

AXIOM R (RATIONALITY). The strict relation $\succ$ on $\mathscr{P}$ is asymmetric and negatively transitive.

Recall that these if $\succ$ is asymmetric and negatively transitive, it is complete and transitive, and therefore rational. I should note that this requirement of rationality is a bit more demanding than the original one, where the alternatives were outcomes instead of lotteries.

The next assumption is one of continuity. It tells us that very small changes in probabilities do not affect the ordering between two lotteries. Intuitively, if there are two lotteries, $p \succ q$, and we take two other lotteries, $\hat{p}$ that is sufficiently close to p and $\hat{q}$ that is sufficiently close to q, then $\hat{p} \succ \hat{q}$.

AXIOM C (CONTINUITY). Let $p, q, r \in \mathscr{P}$ be such that $p \succ q \succ r$. Then there exist $\alpha, \beta \in (0, 1)$, such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.

Axiom C is sometimes called the Archimedean axiom. Intuitively, since p is preferred to q, then no matter how bad r is, we can find some mixture of p and r weighted by α that is close enough to p, so that this mixture is better than q. Similarly, we can find another mixture between the two, this time weighted by β that is close enough to r, so that q is preferred to this new mixture. In other words, the preference relation is *continuous*.

For example, suppose you are interested in money and living. Suppose there are three possible outcomes, you get $\$10^6$, you get nothing, and you die. Consider now the following lotteries, $p = (2/3, 1/3, 0)$, $q = (0, 1, 0)$, and $r = (0, 0, 1)$. You (naturally) have a preference ordering $p \succ q \succ r$. Now, consider the compound lottery $\hat{p} = \alpha p + (1-\alpha)r$ where $\hat{p}(\text{death}) = 1 - \alpha > 0$. Even in this extreme example, continuity requires that we can find $\alpha > 0$ such that $\hat{p} \succ q$: that is, you would accept a strictly positive risk of death (but a chance to get a million bucks) to the lottery in which you stay alive for sure (but get nothing). This α should obviously be pretty large to minimize the risk and make it work. Imagine I told you that your risk of dying on a particular day from driving is 10^{-6}, then you probably would agree that the risk $1 - \alpha < 10^{-6}$ is worth taking: you'd be facing the same risk as you do every day but now you have an almost certain chance to win a million. The point of continuity is to ensure that such a probability always exists no matter how bad the third lottery is. If there was a sudden "jump" in your preferences so that the moment you incur the slightest risk of death, you strictly prefer q, this axiom would be violated.

Axiom C rules out **lexicographic** preferences. Lexicographic preferences are preferences where one of the outcomes has the highest priority in determining the preference ordering. The name comes from the way a dictionary is organized: the first letter of each word has highest priority, and only then the rest follow. For example, suppose that outcomes are defined by the pair (m, c), where m is the make of car I get, and c is its color. Define $x \succeq y$ if either "$[m_x \succ m_y]$" or "$[m_x = m_y] \wedge [c_x \succeq c_y]$". That is, in the preference ordering on the car-color set, the make of car has highest priority. If I prefer one make to another, then the color is irrelevant. If I am indifferent between two makes, only then does color come into play. Although lexicographic preferences are rational, some cannot be represented by a utility function. In particular, if the quantities can be any non-negative real value, these preferences violate continuity: for the decreasing convergent sequence $x_n \to 0$, the preference is $(x_n, 0) \succ (0, 1)$ but at the limit we have $(0, 1) \succ (0, 0)$. In risky choices, the probabilities are real values, so continuity rules out lexicographic preferences.

For me, the real question is whether people *actually* have these types of preferences. I know that people sometimes *claim* that they do, but let's think of what these types of preferences would imply. If one's preferences are lexicographic, then one would not be willing to trade *any* risk of not getting the "main" good for *any* gain on the "secondary" one. For instance, suppose one claimed "give me liberty or give me death." Presumably, this means that one would rather die than accept *any* restriction on one's liberties. Really? While these might be useful for organizing dictionaries, I am not aware of many things in this life that involve infinite preferences of this type. I think one should treat Axiom C as a technical assumption rather than a substantive one.

The third assumption is that if we mix each of two lotteries with a third one, then the preference ordering of the resulting mixtures does not depend on which particular third lottery we used. That is, it is independent of the third lottery.

AXIOM I (INDEPENDENCE). The preference ordering $\succ$ on $\mathcal{P}$ satisfies the *independence axiom* if for all $p, q, r \in \mathcal{P}$ and any $\alpha \in (0, 1)$, the following holds:

$$p \succ q \Leftrightarrow \alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r.$$

This is sometimes called the *Substitution Axiom*. In the compound lottery the player is getting r with the same probability, $(1 - \alpha)$, and thus the "same part" should not affect his preferences. The entire difference should be in how the player evaluates the lotteries p and q against each other. Since $\alpha > 0$, there is a chance that the difference $p \succ q$ will matter, and so the player must prefer the first compound lottery to the second one. This is sometimes called the substitution axiom because one can substitute any third lottery while preserving the ordering of the original two.

Continuing with our example, this requires that adding the same positive risk of death to either of the original "no death" lotteries would not change the preferences: you'd still prefer the (now potentially deadly) lottery that gives you a million bucks to the (now equally potentially deadly) lottery that gives you nothing. This does not depend on how large the risk is. So, suppose $\alpha = .10$, and consider $\hat{p} = 0.10p + 0.90r$ and $\hat{q} = 0.10q + 0.90r$. This axiom ensures that $\hat{p} \succ \hat{q}$: the new lotteries involve the same risk of death but in case you stay alive, $\hat{p}$ would give you the million but $\hat{q}$ would leave you with nothing. That is, the preference between two lotteries p and q should determine which of the two the decision maker prefers to have as a part of a compound lottery regardless of the other possible outcome of the compound lottery, say r. This other outcome r is irrelevant for the choice because r does not occur together with either p or q, but only instead of them.

Although this discussion probably makes Axiom I sounds quite reasonable, it might be among the easiest to violate empirically. For instance, let's look at a variant of the so-called **Allais Paradox**. Consider three possible dollar outcomes: $\mathcal{X} = \{0, 1000, 1100\}$. Suppose I offer you two simple lotteries, p_1 and q_1, which are defined as follows:

$$p_1 = (0.01, 0.66, 0.33) \quad \text{and} \quad q_1 = (0, 1, 0).$$

That is, lottery p_1 gives you \$0 with probability 1%, \$1,000 with probability 66%, and \$1,100 with probability 33%. Lottery q_1, on the other hand, gives you \$1,000 with certainty. What is your preference among these two? (Write it down.)

Now suppose I give you two other lotteries, p_2 and q_2, defined as follows:

$$p_2 = (0.67, 0, 0.33) \quad \text{and} \quad q_2 = (0.66, 0.34, 0).$$

That is, p_2 gives you \$0 with probability 67% and \$1,100 with probability 33%, whereas q_2 gives you \$0 with probability 66% and \$1,000 with probability 34%. What is your preference among these two? (Write it down.)

Did you express a preference $q_1 \succ p_1$? If so, did you also express a preference $p_2 \succ q_2$? (Or, did you express a preference $p_1 \succ q_1$ and $q_2 \succ p_2$?) If you did (and people sometimes do), your stated preferences violate Axiom I. To see that, observe that we can think of p_1 as yielding \$1,000 with 66% and (\$0 with 1% and \$1,100 with 33%). We can also think of q_1 as yielding \$1,000 with 66% and \$1,000 with 34%. Note now that \$1,000 with 66% occurs in both cases. According to the independence axiom, your preference between p_1 and q_1 should not be affected by this because both under p_1 and q_1, you will be getting \$1,000 with 66%. Therefore, if you express a preference $q_1 \succ p_1$, it must be because of the components that differ, so we conclude that you must prefer \$1,000 with 34% to (\$0 with 1% and \$1,100 with 33%).

Turning now to the second set of lotteries, note that we can also think of p_2 as yielding \$0 with 66% and (\$0 with 1% and \$1,100 with 33%). Of course, q_2 yields \$0 with 66% and \$1,000 with 34%. Observe now that \$0 with 66% occurs in both lotteries. If Axiom I holds, this should not affect your preference over these lotteries, which should be entirely determined by your preference over the components that are different. If you expressed a preference $p_2 \succ q_2$, you are saying that you prefer (\$0 with 1% and \$1,100 with 33%) to \$1,000 with 34%. But this contradicts the preference you expressed in the first case! Therefore, independence must be violated.

Some people interpret this to say that people are sensitive to what the third lottery (the one that gets added to the original ones) actually is. For instance, they would say that it matters whether you are adding \$0 with 66% or if you are adding \$1,000 with 66%. I am not sure this is what's going on here. How many of you actually noticed that there is this third lottery that is being added to the other two? How many thought of decomposing the original lotteries the way we did above? My guess is that not many did (when we do this in class, I have yet to see anybody notice this!) In fact, I am pretty sure that if the problem is presented in its "decomposed" form, then people will actually express preferences that will satisfy the independence axiom. What this tells us, then, is that people do not have a strong intuitive feel for lotteries, so they do not evaluate them "properly" (in they way they would if one explained to them what it is that they are actually comparing).

This is a problem when one is interested in the positivist side of game theory; that is, when one wants to explain behavior. The independence axiom is required for preferences to be representable with a function that has the expected utility form, and if someone's preference violate this axiom, then we will not be able to represent their preferences with expected utilities. As a result, we will not be able to explain their behavior through maximization of expected utility. If, on the other hand, the goal is prescriptive (normative), then this is not that big of a problem. If we want to see what the "best" lottery would be, then the independence axiom must be satisfied when we determine the preferences.

Since we are mostly interested in the positivist aspects, the results of the simple exercise above is troubling. Note that it is *not* the case that people always violate independence.

(In informal class experiments with this example, only about 16% seem to do.) The point is that there is a significant number of people that do, and this should give us some pause about the strength of the assumption.

3.3 The Expected Utility Theorem

Recall that we were able to prove that we can represent rational preferences with numbers (under some conditions). We now want to see whether we can do "the same" for the lotteries. Recall that preferences over lotteries cannot be derived from preferences over outcomes at the very least because each individual will evaluate risk subjectively, so two individuals with the same preference ordering over certain outcomes may have very different preferences over lotteries involving these outcomes. Given then the preference ordering over lotteries, we want to be able to represent this numerically. There are numerous ways one can choose to map preferences into numbers but the *expected utility* functional form is especially convenient (mathematically) and quite intuitive. With this form, you assign numbers (utilities) to the outcomes, as before, and then calculate the expected utility of a lottery by taking the probability with which an outcome occurs in this lottery and multiplying it by the utility of this outcome, then summing over all outcomes in the support of the lottery. The question is whether given preferences over lotteries we can guarantee that we can find numbers that would make this calculation work such that the ranking of the expected utilities of two lotteries will be the same as their preference ordering. Let's first formally define the function we are interested in using.

DEFINITION 6. Let X denote a finite set of outcomes. The utility function $U : \mathcal{P} \to \mathbb{R}$ has the **expected utility form** if there is a (Bernoulli) payoff function $u : X \to \mathbb{R}$ that assigns real numbers to outcomes such that for every simple lottery $p \in \mathcal{P}$, we have

$$U(p) = \sum_{x \in X} p(x)u(x).$$

A utility function with the expected utility form is called a **von Neumann-Morgenstern (vNM) expected utility function**.[2]

In other words, the utility function U takes a lottery p and returns a real number, say, α_p. This function has the expected utility form if we could assign payoffs to the outcomes, $u(x)$, such that when we compute the expected payoff of the lottery p, we get α_p; that is, we can find $u(x)$ such that $\sum_{x \in X} p(x)u(x) = \alpha_p$. This is the definition of a vNM function. What we want to know is whether such a function can represent the preferences over the risky choices. The theorem we now prove says that we can provided we make some assumptions about these preferences.

The *Expected Utility Theorem* first proved by von Neumann in 1944 is the cornerstone of game theory. It states that if the decision-maker's preferences over lotteries satisfy Axioms R, C, and I, then these preferences are representable with a function that has the

[2] Although this is called the vNM expected utility representation, it is much older, going back to Daniel Bernoulli in the 18th century. Nowadays, vNM is used for expected utility functions, and Bernoulli usually refers to payoffs for certain outcomes. If the set of outcomes is a continuum (a, b), then the lottery is a probability density function f defined on (a, b), and the vNM function is $U(p) = \int_a^b f(x)u(x)\mathrm{d}x$.

expected utility form. That is, if the axioms of rationality, continuity, and independence are accepted, then we can assign real numbers to the outcomes such that the relationship between the expected utilities of any two lotteries preserves their rank ordering. In other words, we can move from using preference orderings to using numbers (utilities), which opens up the entire mathematical arsenal for analysis.

Let's make sure we understand what exactly it is that we are after here. We know that when we are dealing with rational preferences over certain outcomes, we can represent the outcomes with numbers that preserve the preference ordering over the outcomes. However, in many cases we will be dealing with risky choices that involve uncertain outcomes. We saw how to represent this situation with lotteries. And now we are going to see how to represent the preferences over these lotteries with numbers that are produced by calculating expected utilities of these lotteries.

For example, suppose that each day I leave home to come to my office, I face three possible outcomes: getting to work safely, having a minor accident, and having a major accident. Suppose that I have two ways to get to work by driving either a scooter or a car. Each mode of transportation is associated with a lottery over these three outcome. I ride the scooter very carefully because I have to be alert about drivers not seeing me (two accidents already!). So, the chance of an accident is smaller than when I drive the car (because I am what you call an aggressive driver). However, there's no margin for errors on the scooter, so the chance of a major accident is higher than when I drive the car. So, suppose the following two lotteries summarize all of these probabilities: $d = (0.87, 0.12, 0.01)$ if a drive the car, and $r = (0.94, 0.04, 0.02)$ if I ride the scooter. That is, the chances of getting safely to work are higher on the scooter (94% versus 87%) and the chances of getting into a minor accident are lower (4% versus 12%). However, the chance of a major accident is higher (2% versus 1%). Since I ride the scooter every day, it must be the case that $r \succ d$. The question now is, can we find numbers (u_1, u_2, u_3) such that when we assign them to the outcomes (safe, minor accident, major accident) and calculate the expected utilities, we would get $U(r) > U(d)$ (that is, preserve the preference ordering)?

The following theorem tells us that it is possible. Therefore, we can assign numbers to outcomes and then calculate the expected utility of a lottery in the way we know how. The ordering of the expected utilities preserves the preference ordering of the lotteries. The decision-maker chooses the lottery that yields highest expected utility because this is his most preferred lottery, *an extremely powerful result*. In particular, it allows us to assign payoffs over the outcomes and then *infer* the preferences over the lotteries by assuming some attitudes toward risk: computing the expected utilities using these payoffs would rank order the risky choices and provide the basis for forming expectations about the choices the agent would make.

THEOREM 1 (EXPECTED UTILITY THEOREM). *A preference relation $\succ$ on the set $\mathcal{P}$ of simple lotteries on X satisfies Axioms R, C, and I if, and only if, there exists a function that assigns a real number to each outcome, $u : X \to \mathbb{R}$, such that for any two lotteries $p, q \in \mathcal{P}$, the following holds:*

$$p \succ q \Leftrightarrow \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)u(x).$$

□

The equation stated in the theorem reads "a lottery p is preferred to lottery q if, and only if, the expected utility of p is greater than the expected utility of q." Obviously, to calculate the expected utility of a lottery, we must know the utilities attached to the actual outcomes, that is, the number that $u(x)$ assigns to outcome x.

The theorem makes two claims. First, it states that if the preference relation $\succ$ can be represented by a vNM utility function, then must satisfy Axioms R, C, and I. This is the necessity (only if) part of the claim, which we won't prove here. Second, and more importantly, the theorem states that if the preference relation satisfies Axioms R, C, and I, then must be representable by a vNM utility function. This is the sufficiency (if) part of the claim, to whose proof we now turn.

We begin by assuming that $\succ$ satisfies Axioms R, C, and I. We now want to find a utility function $U : \mathcal{P} \to \mathbb{R}$ such that $p \succ q \Leftrightarrow U(p) > U(q)$, and this function has the expected utility form. The proof proceeds in several steps, where we show that:

1. We can calibrate any lottery as follows. For each lottery p, we can always find another (calibrated) lottery c_p in which the best outcome occurs with probability $\alpha_p \in [0, 1]$ and the worst outcome occurs with probability $1 - \alpha_p$, such that the player is indifferent between p and c_p. That is, for any arbitrary lottery, we can find another lottery that involves only the best and worst outcomes such that the player is indifferent between the two.

2. The number α_p that calibrates c_p to p is unique, so the calibration is unique: each lottery p has its own unique α_p.

3. In simple lotteries over the best and worst outcomes, the decision-maker always prefers those that yield the best outcome with higher probability. So, if in the lottery c_p the best outcome occurs with probability α_p (and the worst with probability $1 - \alpha_p$) and in lottery c_q the best outcome occurs with probability $\alpha_q < \alpha_p$ (and the worst with probability $1 - \alpha_q$), then it must be the case that $c_p \succ c_q$. The converse also holds: if c_p and c_q are lotteries that involve only the best and worst outcomes and $c_p \succ c_q$, then $\alpha_p > \alpha_q$.

4. By the consequentialist premise, the player does not care whether he is comparing lotteries p and q or their calibrated equivalents c_p and c_q. Thus, $c_p \succ c_q \Leftrightarrow p \succ q$.

5. But since $c_p \succ c_q \Leftrightarrow \alpha_p > \alpha_q$, this now means that $p \succ q \Leftrightarrow \alpha_p > \alpha_q$. That is, the calibration numbers represent the preferences over the lotteries, so we can use $U(p) = \alpha_p$ for any p. This establishes that preferences can be represented with a function $U : \mathcal{P} \to \mathbb{R}$; that is, $p \succ q \Leftrightarrow U(p) > U(q)$.

6. Finally, we show that this function U has the expected utility form. That is, we can assign numbers $u(x)$ to the outcomes $x \in X$ such that for any lottery p, $U(p) = \sum_{x \in X} p(x)u(x) = \alpha_p$.

The fundamental insight in the proof is that we can associate each lottery with a special "magic" number, which is the weight on the best outcome in a lottery over the best and worst outcomes only. This is the number that the function U would assign, and it is also the number that one should get when one computes the expected utility of the lottery. On now to the actual proof.

Proof. Because the set of outcomes X is finite (and has at least two elements or else there is no choice to be made), it follows that there must be a best outcome, b, and a worst outcome w. Now let p^b denote the degenerate lottery that yields b with probability 1, and p^w be the degenerate lottery that yields w with probability 1. Then, for any lottery $p \in \mathcal{P}$, we have $p^b \succeq p$ and $p \succeq p^w$. If $p^b \sim p^w$, the conclusion in the theorem follows trivially because this implies that the player is indifferent among any two lotteries in $\mathcal{P}$. Therefore, from now on, we assume that $p^b \succ p^w$.

We begin by showing that we can *calibrate* the player preference for any lottery in terms of a lottery involving only the best and worst outcomes. That is, for every lottery p, there is another lottery that puts positive probability only on the best and the worst outcomes such that the player is indifferent between p and that other lottery. Furthermore, we will show that the probability with which the best outcome occurs in this calibrated lottery is unique. Formally,

CLAIM 1. *For any $p \in \mathcal{P}$, there is a unique $\alpha_p \in [0,1]$ such that $p \sim \alpha_p p^b + (1-\alpha_p) p^w$.* □

The existence of the calibrated lottery $c_p \equiv \alpha_p p^b + (1-\alpha_p) p^w$ follows from Axiom C and the fact that $p^b \succ p^w$. That's because continuity of $\succ$ implies continuity of $\sim$, so Axiom C can be restated for $\sim$ as follows: for any $p, q, r \in \mathcal{P}$, there exists $\alpha \in [0,1]$ such that $q \sim \alpha p + (1-\alpha) r$.

We now want to show that α_p is unique. We shall prove this by contradiction. Suppose that there exist two numbers α and β such that $p \sim \alpha p^b + (1-\alpha) p^w$ and $p \sim \beta p^b + (1-\beta) p^w$. To simplify notation, let $c_a \equiv \alpha p^b + (1-\alpha) p^w$ and $c_b \equiv \beta p^b + (1-\beta) p^w$. Suppose without loss of generality that $\alpha > \beta$. We now show that this implies $c_a \succ c_b$. Note that we can write $c_a = \gamma p^b + (1-\gamma) c_b$, where $\gamma = \frac{\alpha-\beta}{1-\beta}$. Observe that because $\alpha, \beta \in (0,1)$ and $\alpha > \beta$, we know that $\gamma \in (0,1)$.[3] Because $\gamma > 0$ and p^b is the most preferred lottery (since it gives the best outcome for sure), it follows that $\gamma p^b + (1-\gamma) c_b \succ c_b$. Intuitively, we are comparing c_b to a lottery that involves c_b itself and the best lottery p^b with positive probability, so the latter must be better. To see this formally, we use Axiom I. First, observe that $p^b \succ c_b = \beta p^b + (1-\beta) p^w$. This is because Axiom I implies that when $p^b \succ p^w$, we have $p^b = \beta p^b + (1-\beta) p^b \succ \beta p^b + (1-\beta) p^w = c_b$. (Notice that βp^b occurs in both cases, so the preference between the two lotteries is entirely determined by $p^b \succ p^w$.) Now observe that because $p^b \succ c_b$, Axiom I also implies that $\gamma p^b + (1-\gamma) c_b \succ \gamma c_b + (1-\gamma) c_b = c_b$. (Notice that $(1-\gamma) c_b$ occurs in both cases, so the preference between the two lotteries is entirely determined by $p^b \succ c_b$.)

We conclude that $c_a \succ c_b$. But this contradicts $p \sim c_a$ and $p \sim c_b$: a player cannot be indifferent between two lotteries, one of which he strictly prefers to the other! Therefore, $\alpha = \beta = \alpha_p$, which means that the number α_p is unique.

[3] Quick check using the definition of γ so $1-\gamma = (1-\alpha)/(1-\beta)$:

$$\frac{\alpha-\beta}{1-\beta} p^b + \frac{1-\alpha}{1-\beta} c_b = \frac{\alpha-\beta}{1-\beta} p^b + \frac{1-\alpha}{1-\beta}\left[\beta p^b + (1-\beta) p^w\right] = \alpha p^b + (1-\alpha) p^w = c_a.$$

To gain some intuition about the calibration process, consider the following thought experiment. Take some lottery p that yields neither the best nor the worst outcome with certainty. Clearly, you must prefer the best outcome with certainty to this lottery: $p^b \succ p$. Analogously, you must prefer this lottery to the worst outcome with certainty: $p \succ p^w$. Consider now the compound lottery $\alpha p^b + (1-\alpha)p^w$. If α is close enough to 1, then the compound lottery is only slightly less preferable than p^b (this follows from Axiom C). Thus, it must still be preferable to p too. As you begin to decrease α, the compound lottery becomes less and less attractive because the likelihood of the worst outcome begins to increase. The lottery gets closer and closer to p and at some point you will be precisely indifferent between that lottery and p itself. The α at which this occurs is the calibration number α_p for lottery p. If you decrease α further, the compound lottery will become strictly worse than before, and therefore p will be strictly preferable to it. It should be clear from this thought experiment that each arbitrary lottery p has a unique calibration number associated with it.

We have now reached the crucial point: we have shown that any arbitrary lottery can be replaced by a lottery which assigns positive probabilities only to the best and worst outcomes. By the consequentialist premise, the decision-maker would not care whether he faces p or the equivalent calibrated lottery $c_p = \alpha_p p^b + (1-\alpha_p)p^w$. But since the calibration number α_p is unique, we can define a function U that takes a lottery and returns its associated calibration number. We claim that this function represents the preference ordering. Formally,

CLAIM 2. *The function $U : \mathcal{P} \to \mathbb{R}$ that assigns $U(p) = \alpha_p$, where α_p is such that $p \sim \alpha_p p^b + (1-\alpha_p)p^w$, for all $p \in \mathcal{P}$ represents the preference relation $\succ$. That is,*

$$p \succ q \Leftrightarrow U(p) > U(q).$$

□

To prove this, note that the function U takes a lottery and returns its unique calibration number, which is the probability with which the best outcome occurs in the simple calibrated lottery. But in such simple lotteries that involve the best and worst outcomes only, the player must prefer those that yield the best outcome with higher probability. Formally,

CLAIM 3. *For any $\alpha, \beta \in (0,1)$, $\alpha p^b + (1-\alpha)p^w \succ \beta p^b + (1-\beta)p^w$ if, and only if, $\alpha > \beta$.* □

We have to prove both parts of the claim. Recall that we use c_a and c_b to simplify notation.

(Sufficiency.) Assume $\alpha > \beta$. We have already shown that $c_a = \gamma p^b + (1-\gamma)c_b \succ c_b$ where $\gamma = (\alpha-\beta)/(1-\beta) \in (0,1)$ in the argument above, so this part holds.

(Necessity.) Assume $c_a \succ c_b$ and suppose, seeking contradiction, that $\alpha \leq \beta$. If $\alpha = \beta$, then $c_a \sim c_b$, a contradiction. If $\alpha < \beta$, then applying the previous argument with γ (and reversing α and β) leads to the conclusion that $c_b \succ c_a$, a contradiction. Therefore, $\alpha > \beta$ must hold.

Take now any two lotteries $p, q \in \mathcal{P}$ and their associated calibration numbers α_p and α_q. From the definition of these numbers and Claim 3, we know that $p \sim \alpha_p p^b + (1-\alpha_p)p^w \succ \alpha_q p^b + (1-\alpha_q)p^w \sim q$ if, and only if, $\alpha_p > \alpha_q$. Thus, taking $U(p) = \alpha_p$ for any $p \in \mathcal{P}$ represents the preferences!

We have now concluded that we can represent preferences over lotteries with an unknown function U which maps lotteries into real numbers in a well-defined way. We now want to show that this function $U(p) = \alpha_p$ has the expected utility form. That is, we want to show that there exists an assignment of payoffs to outcomes, $u(x)$, such that for any lottery p, $\sum_{x \in X} p(x)u(x) = \alpha_p$. This part of the proof requires two steps. First, we show that $U(p) = \alpha_p$ is linear. Next, we show that a function has the expected utility form if, and only if, it is linear.[4]

CLAIM 4. *The utility function $U(p) = \alpha_p$ is linear.* □

The definition of a linear function means that for any $p, q \in \mathcal{P}$ and $\alpha \in [0, 1]$, the following must hold: $U(\alpha p + (1-\alpha)q) = \alpha U(p) + (1-\alpha)U(q)$. We want to show that it holds for $U(p) = \alpha_p$. By the definition of α_p, we know that $p \sim \alpha_p p^b + (1-\alpha_p)p^w$, and using $U(p) = \alpha_p$, we can rewrite this as $p \sim U(p)p^b + (1-U(p))p^w$. Analogously, we can write $q \sim U(q)p^b + (1-U(q))p^w$. Using these definitions, we can now write:

$$\alpha p + (1-\alpha)q \sim \alpha \left[U(p)p^b + (1-U(p))p^w \right] + (1-\alpha) \left[U(q)p^b + (1-U(q))p^w \right]$$

from which we can collect terms on p^b and p^w to get:

$$\equiv [\alpha U(p) + (1-\alpha)U(q)]\, p^b + [1 - (\alpha U(p) + (1-\alpha)U(q))]\, p^w$$

or, letting $\alpha_r = \alpha U(p) + (1-\alpha)U(q)$, we can rewrite this as:

$$\equiv \alpha_r p^b + (1-\alpha_r)p^w.$$

Since we now found that $\alpha p + (1-\alpha)q \sim \alpha_r p^b + (1-\alpha_r)p^w$, we know that by definition, $U(\alpha p + (1-\alpha)q) = \alpha_r$. That is, we found the unique calibration number of the compound lottery $\alpha p + (1-\alpha)q$ we are examining. But since $\alpha_r = \alpha U(p) + (1-\alpha)U(q)$ from our own definition above, we now have:

$$U(\alpha p + (1-\alpha)q) = \alpha_r = \alpha U(p) + (1-\alpha)U(q),$$

which is what we wanted to prove. That is, $U(\cdot)$ is linear. We now need to show that a function has the expected utility form if, and only if, it is linear. Formally,

CLAIM 5. *A utility function $U : \mathcal{P} \to \mathbb{R}$ has the expected utility form if, and only if, it is linear.* □

[4]A linear function $f(\cdot)$ satisfies two properties: (i) additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$, and (ii) homogeneity: $f(\alpha x) = \alpha f(x)$. More generally, for any integer $K > 0$, the function f is linear if $f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_K x_K) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_K f(x_K)$. Writing this in compact form tells us that $f(\cdot)$ is linear if:

$$f\left(\sum_{k=1}^{K} \alpha_k x_k\right) = \sum_{k=1}^{K} \alpha_k f(x_k).$$

Recall that p^x is the degenerate lottery that assigns probability 1 to the outcome $x \in X$. We can rewrite any $p \in \mathcal{P}$ as a combination of degenerate lotteries as follows:

$$p = \sum_{x \in X} p(x) p^x.$$

Note that this involves vector addition.[5]

(Sufficiency). Assume that $U(\cdot)$ is linear, and show that it must have the expected utility form. Because U is linear (in the probabilities), this implies that:

$$\begin{aligned}
U(p) &= U\left(\sum_{x \in X} p(x) p^x\right) && \text{by the equivalent definition of } p \\
&= \sum_{x \in X} U(p(x) p^x) && \text{by the additivity of } U \\
&= \sum_{x \in X} p(x) U(p^x) && \text{by the homogeneity of } U \\
&= \sum_{x \in X} p(x) u(x) && \text{by the definition of } U\text{: } U(p^x) = u(x),
\end{aligned}$$

and so $U(\cdot)$ has the expected utility form.

(Necessity.) Assume that $U(\cdot)$ has the expected utility form, and show that it is linear. Take any compound lottery p that consists of K simple lotteries: $p = (p_1, p_2, \ldots, p_K; \alpha_1, \alpha_2, \ldots, \alpha_K)$. Recall that we can reduce p to the simple lottery $\sum_{k=1}^{K} \alpha_k p_k$. Note that this involves vector addition: $p_k(x)$ is the probability that the simple lottery p_k assigns to outcome $x \in X$. We can now write:

$$\begin{aligned}
U\left(\sum_{k=1}^{K} \alpha_k p_k\right) &= \sum_{x \in X} \left[\sum_{k=1}^{K} \alpha_k p_k(x)\right] u(x) && \text{by the expected utility form of } U \\
&= \sum_{k=1}^{K} \alpha_k \left[\sum_{x \in X} p_k(x) u(x)\right] && \text{algebraic manipulation} \\
&= \sum_{k=1}^{K} \alpha_k U(p_k) && \text{by the expected utility form of } U.
\end{aligned}$$

This establishes that $U(\cdot)$ is linear, and completes the proof. ■

[5] To see what this statement means, suppose there are $N = 3$ outcomes and let $p = (1/3, 1/2, 1/6)$ be some simple lottery. The three degenerate lotteries are $p^1 = (1, 0, 0)$, $p^2 = (0, 1, 0)$, and $p^3 = (0, 0, 1)$. Defining p in terms of these degenerate lotteries is easy:

$$\sum_{i=1}^{N} p(x_i) p^i = 1/3 p^1 + 1/2 p^2 + 1/6 p^3 = (1/3, 0, 0) + (0, 1/2, 0) + (0, 0, 1/6) = (1/3, 1/2, 1/6) = p.$$

The proof shows that it is possible to assign numbers to the outcomes such that the preferences are representable by a function with the expected utility form. The proof is constructive and shows *one* such possibility: $U(p) = \alpha_p$. But since $\alpha_p \in [0,1]$ and $U(\cdot)$ has the expected utility form, it follows that $u(x) \in [0,1]$ for all $x \in X$. A little bit of thought tells us that $u(b) = 1$ and $u(w) = 0$. To see this, take any two lotteries $p, q \in \mathcal{P}$ such that $p^b \succ p \succ q \succ p^w$, and note that the definition of $U(\cdot)$ and the fact that it has the expected utility form yield:

$$U(p) = \alpha_p U(p^b) + (1-\alpha_p)U(p^w) = \alpha_p u(b) + (1-\alpha_p)u(w) = \alpha_p$$
$$U(q) = \alpha_q U(p^b) + (1-\alpha_q)U(p^w) = \alpha_q u(b) + (1-\alpha_q)u(w) = \alpha_q.$$

The following system of equations that must therefore be satisfied:

$$\alpha_p u(b) + (1-\alpha_p)u(w) = \alpha_p$$
$$\alpha_q u(b) + (1-\alpha_q)u(w) = \alpha_q.$$

From the second equation, we get $u(w) = \alpha_q(1-u(b))/(1-\alpha_q)$, which we can do because from $p \succ q$ we know that $1 > \alpha_p > \alpha_q > 0$. Using this in the first equation then produces:

$$\left[\frac{(1-\alpha_p)\alpha_q}{\alpha_p(1-\alpha_q)}\right](1-u(b)) = (1-u(b)).$$

Since $\alpha_p, \alpha_q \in (0,1)$, the bracketed coefficient on the left-hand side is strictly positive. Therefore, there is only one solution to that equation, and it is $u(b) = 1$ (which is why you can't divide both sides by $1 - u(b)$). This, in turn, pins down $u(w) = 0$. The construction in the proof requires that the numbers $u(x)$ are *always* be between zero and one.

This is a feature of the algorithm and in this sense the utilities over outcomes that it assigns are unique. But this is only one possible construction of these utilities. It is sufficient to demonstrate existence of such an assignment, which is what the theorem claims. But there are other, infinitely many, assignments that would work just as well. This is a consequence of the utility function $U(\cdot)$ being linear and the fact that its properties are preserved by increasing linear transformations.

PROPOSITION 4. *If $U : \mathcal{P} \to \mathbb{R}$ is a vNM expected utility function, then $\hat{U} : \mathcal{P} \to \mathbb{R}$ is another vNM expected utility function if, and only if, there are scalars $a > 0$ and b such that:*

$$\hat{U}(p) = aU(p) + b$$

for every $p \in \mathcal{P}$. □

The expected utility property is a *cardinal* property of utility functions defined on the space of lotteries. Because $U(\cdot)$ is linear, this property is preserved by the type of transformations shown in Proposition 4. This result tells us that we can generate any number of equivalent expected utility functions by such linear transformations, and in this sense the assignment of utilities over outcomes is not unique.

That is, from the constructive proof of the expected utility theorem we know how to find one particular $U(\cdot)$ that represents the preferences, and that it involves $u(b) = 1$ and

$u(w) = 1$, with $u(x) \in [0, 1]$ for any $x \in X$. But from Proposition 4, we know that from this $U(\cdot)$ we can construct any number of alternative expected utility functions that will also represent these preferences. In these functions, $u(x)$ will not have to be confined to [0, 1].

An important consequence of Proposition 4 is that for vNM expected utility functions, *differences in utilities have meaning* because they imply the preferences over lotteries. For example, suppose there are four outcomes with utilities $u(x_i) = u_i$. We would write the statement "the difference in utility between outcomes x_1 and x_2 is greater than the difference in utility between outcomes x_3 and x_4" as $u_1 - u_2 > u_3 - u_4$, which is equivalent to:

$$1/2u_1 - 1/2u_2 > 1/2u_3 - 1/2u_4 = 1/2u_1 + 1/2u_4 > 1/2u_2 + 1/2u_3.$$

This would now imply that the lottery $p = (1/2, 0, 0, 1/2)$ must be preferred to the lottery $q = (0, 1/2, 1/2, 0)$. Since preferences over outcomes are ordinal, this is another reason why we cannot derive preferences over lotteries from them but must specify them as part of the description. That is, it is not the case that we are suddenly introducing cardinal considerations into an ordinal world. Instead, we begin by saying $p \succ q$, and then finding $u(x)$ such that $U(p) > U(q)$. The ranking is preserved by all linear transformations of the vNM utility function.

It is worth emphasizing that it is incorrect to say that a decision-maker prefers an outcome x_1 over outcome x_2 because the utility of x_1 is higher than the utility of x_2, or $u(x_1) > u(x_2)$. Rather, *because the decision-maker prefers x_1 to x_2*, the utilities that represent these outcomes are such that $u(x_1) > u(x_2)$. Similarly, a decision-maker does not prefer a lottery p to lottery q because the expected utility of p is higher than the expected utility of q. Instead, it is because he prefers p to q that $U(p) > U(q)$. People do not have utilities and they do not maximize expected utilities. They have preferences over uncertain choices and we, as analysts, find it convenient to represent these preferences with expected utilities. There are infinite ways in which we can represent these preferences actually, including infinite variations by going so with expected utility functions. This is frequent confusion (even in print), which you have to avoid scrupulously. Remember that utility functions are abstract entities that use fictitious numerical values to represent preferences over risky alternatives. The preference orderings are fundamental, and the utility functions are their (non-unique) representations only. It is worth stating this emphatically, *game theory does **not** assume that people are expected-utility maximizers!* All we assume is that people will take actions that they judge most likely to yield the outcomes they most prefer, however the evaluate these outcomes internally. The numbers are purely representational and introduced so we can analyze these decisions using mathematical tools.

3.4 How to Think about Expected Utilities

3.4.1 Ride'n'Maim Example

Recall my ride-and-maim scenario. The best outcome is "arrive safely", so the degenerate lottery p^b assigns this probability 1. The worst outcome is "major accident," so the degenerate lottery p^w assigns that probability 1. We now want α_d such that $d \sim \alpha_d p^b + (1 - \alpha_d) p^w$; that is, we want the appropriate mixture between the best and worst lotteries that makes me indifferent between that compound lottery and my original lottery associated with driving the car. Similarly, we want α_r such that $r \sim \alpha_r p^b + (1-\alpha_r)p^w$; that is, the appropriate mixing between the best and worst lotteries that makes me indifferent between the resulting compound lottery and the one associated with riding the scooter. (We know that these must always exist by the continuity axiom.) In my particular case, I am indifferent between a 0.04 risk of a minor accident and a 0.01 risk of a major accident. Recall the original lotteries are $d = (0.87, 0.12, 0.01)$ and $r = (0.94, 0.04, 0.02)$. Since I can trade a 4% probability of minor accident for 1% probability of major accident, this means that I am indifferent between a lottery that decreases the risk of a minor accident by 4% at the cost of increasing the risk of major accident by 1%. That is, $r \sim \hat{r} = (0.97, 0, 0.03)$. To see how I obtained $\hat{r}$, observe that it assigns 4% less chance of a minor accident than r (and since the original risk was only 4%, this makes the probability of that outcome under $\hat{r}$ precisely zero), but at the same time increases the risk of a major accident by 1% (and since original risk was 2%, the new one is 3%). Since $\hat{r}$ is a valid probability distribution, the probabilities it assigns to all outcomes must sum to one, which implies that it must assign a 97% chance to the safe outcome. To derive $\hat{d}$, observe that it involves a 12% risk of a minor accident. Given my preferences, I can trade this for 3% increase in the risk of a major accident. Hence, $d \sim \hat{d} = (0.96, 0, 0.04)$ by analogous logic. The new lotteries involve only the best and worst outcomes, which means we can write $r \sim 0.97p^b + 0.03p^w$ and $d \sim 0.96p^b + 0.04p^w$. That is, we now have the expected utility numbers: $U(r) = \alpha_r = 0.97$ and $U(d) = \alpha_d = 0.96$. Observe that $U(r) > U(d)$, as required by $r \succ d$, which we knew to be my preference. Not surprisingly, given our proof, U represents these preferences by assigning $\alpha_r > \alpha_d$.

At this point, we have the expected utilities given by U but note that we have not assigned any utilities to the outcomes themselves. That is, while we have $U(p)$, we do not yet have (u_1, u_2, u_3), the utilities for (safe,minor,major), respectively. The fact that U is linear means we can assign these numbers. So let's find a set that works. For simplicity of notation, let (x_1, x_2, x_3) denote the set of three outcomes (so x_1 denotes the safe outcome, x_2 the minor accident, and x_3 the major accident). Since U has the expected utility form, we know that:

$$\begin{aligned}
U(r) &= r(x_1)u_1 + r(x_2)u_2 + r(x_3)u_3 = \alpha_r \\
U(d) &= d(x_1)u_1 + d(x_2)u_2 + d(x_3)x_3 = \alpha_d \\
U(\hat{r}) &= 0.97u_1 + 0.03u_3 = \alpha_r \\
U(\hat{d}) &= 0.96u_1 + 0.04u_3 = \alpha_d.
\end{aligned}$$

Substituting the probabilities from the lotteries and the values for α_r and α_d, we obtain (after multiplying both sides of each equation by 100 to simplify the expressions):

$$\begin{aligned}
94u_1 + 4u_2 + 2u_3 &= 97 \\
87u_1 + 12u_2 + 1u_3 &= 96 \\
97u_1 + 3u_3 &= 97 \\
96u_1 + 4u_3 &= 96.
\end{aligned}$$

From the last two equations it immediately follows that $u_3 = 0$ and $u_1 = 1$. We already saw that calculating α_r and α_d must guarantee that the best outcome will always have to be assigned 1 and the worst 0. Using that fact, it becomes very easy to find u_2. The first two equations produce:

$$94 + 4u_2 = 97$$
$$87 + 12u_u = 96.$$

Solving either one gives the same result, $u_2 = 0.75$. Thus, we obtain the assignment of u_x to outcomes that guarantees that preferences over lotteries are represented by a function with the expected utility form. This assignment is $(1, 0.75, 0)$. Observe that these utilities are reasonable in the sense that they also represent the ranking of outcomes: $x_1 \succ x_2 \succ x_3$.

The assignment is unique if we want to keep $u_1 = 1$ and $u_2 = 0$. However, there are infinite ways to generate equivalent expected utility functions that will represent my preferences if we relax that requirement. For example, consider $\hat{U}(p) = 3U(p) - 1$. Since both functions have the expected utility form, we get $\hat{U}(p^b) = 3U(p^b) - 1 = 3u_b - 1$, which implies $\hat{u}_1 = 3(1) - 1 = 2$. Similar calculations give us $\hat{u}_2 = 1.25$ and $\hat{u}_3 = -1$. We now have an assignment $(2, 1.25, -1)$ that I claim also represents the preferences. Let's check:

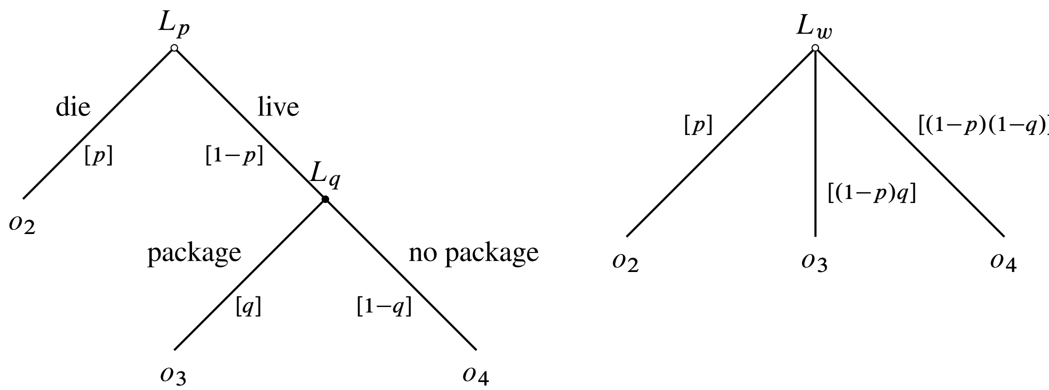
$$\hat{U}(r) = 0.94(2) + 0.04(1.25) + 0.02(-1) = 1.91$$
$$\hat{U}(d) = 0.87(2) + 0.12(1.25) + 0.01(-1) = 1.88.$$

Clearly, $\hat{U}(r) > \hat{U}(d)$, as required. Let's find what probabilities we'd have to assign to the best outcome in the equivalent best/worst lotteries. Since I am indifferent it follows I should get the same expected utility from the lottery r and the equivalent lottery $\hat{r}$, so $\alpha_r(2) + (1 - \alpha_r)(-1) = 1.91 \Rightarrow \alpha_r = 0.97$. That we obtain absolutely the same probability as we started with should not be surprising: after all, the claim is that $\hat{U}$ represents the preferences, so it would have to keep the same probabilities that make me indifferent between r and $\hat{r}$. A final check reveals the same for d: $\alpha_d(2) + (1 - \alpha_d)(-1) = 1.88 \Rightarrow \alpha_d = 0.96$, just as we thought. Hence, the assignment $(2, 1.25, -1)$ does represent the preferences too. It is in this sense that there are infinite ways to do so. As we have seen, they are all intimately related to the basic assignment we derived in the theorem.

3.4.2 An Example with Multiple Outcomes

Consider the following scenario. I have to choose whether to go to work or stay at home on a particular day. If I go to work, I will earn $500 for the day and if I stay home, I earn nothing. If I go to work, there is a 1 in 100,000 chance that I will be killed in a car accident, and if I stay home this risk is 0. Finally, I am expecting a shipment from UPS and this is their last attempted delivery. If I go to work, there is a 80% chance that UPS will attempt delivery while I am at work and I will lose the package, and there is a 20% chance that UPS will attempt delivery after I get home and I will get it. If I stay home, I am certain to get the package.

Let us begin by representing the risky choices as lotteries. Going to work gets me killed with probability $p = 0.00001$ and lets me live with probability $1 - p$. If I die, I will not earn any money and there will be no UPS delivery (because I will not be home even after work). Hence, getting killed yields the outcome \$0, no package, and death, which we shall denote by o_2. If I survive the commute, I will certainly earn \$500 and I will also get the package with probability $q = 0.20$. Let o_3 denote the outcome of getting \$500, receiving the package, and living; let o_4 denote the outcome of getting \$500, not receiving the package, and living. We can represent the choice of going to work as a compound lottery, as illustrated in Figure 3(a). By the consequentialist premise, we can reduce this compound lottery to a simple lottery, L_w, as shown in Figure 3(b). Staying home means getting \$0, receiving the package, and not being killed in a car accident. Let o_1 denote this outcome. Because staying home yields this outcome with probability 1, it can be represented with the degenerate lottery L_h.



(a) Compound Lottery (b) Reduced Simple Lottery

Figure 3: The Risky Choice of Going to Work.

Let (o_1, o_2, o_3, o_4) denote the ordered set of all possible outcomes, and let $L_k(o_i)$ denote the probability assigned by lottery L_k to the outcome o_i. Using short-hand notation we can write the lottery $L_h = (1, 0, 0, 0)$, and the lottery $L_w = (0, p, (1-p)q, (1-p)(1-q))$. Let L_i represent the degenerate lottery that assigns probability 1 to the outcome o_i. For instance, L_2 is the lottery which gives me death, no money and no package with certainty.

We now turn to the preference ordering for the certain outcomes. My least preferred outcome is the one with dying, o_2, and my most preferred outcome is to live, and get both \$500 and the package, o_3. Since the package is not particularly important to me, I also prefer to get \$500 even if it means losing the package, so I prefer o_4 to o_1. Hence, my complete transitive preference ordering over the certain outcomes is:

$$o_3 \succ o_4 \succ o_1 \succ o_2.$$

Because I went to work when I was faced with the choice, I also know that I prefer L_w to L_h as well:

$$L_w \succ L_h.$$

The question is: *how do we assign utilities to the certain outcomes that preserve that preference ordering over the lotteries when we compute the expected utilities of these lotteries?*

We begin by assigning utility 1 to the best outcome and 0 to the worst outcome:

$$u(o_3) = 1 \text{ and } u(o_2) = 0.$$

Of course, this is arbitrary but, as we shall see soon, it is a useful place to start. We can, in fact, use any anchors we wish here as long as $u(o_3) > u(o_2)$. We shall see how to transform the utilities we assigned if we wish to anchor the end-points at different values. Because degenerate lotteries assign zero probability to all outcomes except the one they deliver with certainty, it immediately follows that the expected utilities of L_3 and L_2 are 1 and 0, respectively:

$$U(L_3) = \sum_{i=1}^{4} L_3(o_i)u(o_i) = L_3(o_3)u(o_3) = (1)(1) = 1$$
$$U(L_2) = \sum_{i=1}^{4} L_2(o_i)u(o_i) = L_2(o_2)u(o_2) = (1)(0) = 0.$$

The utilities assigned to outcome o_1 and o_4 are not relevant for these calculations because the terms that involve drop out by being multiplied by zero. However, we obviously have to assign utilities to these outcomes before we can compute the expected utility of any other lottery.

Consider now the best and worst outcomes, and a compound lottery, L_α, that yields L_3 with probability α and L_2 with probability $1 - \alpha$. Obviously, this is equivalent to saying that it leads to outcome o_3 with probability α and outcome o_2 with probability $1-\alpha$. Recall that the best outcome is ($500, life, package), whereas the worst outcome is ($0, death, no package). Take outcome o_1 ($0, life, package), and let us see whether we can find some value for α such that I would be indifferent between L_α and L_1, the lottery that gives me o_1 with certainty.

Recall that I know that I prefer L_w to L_1 already. In other words, I am willing to take some risks in order to get $500. As I have said at the outset, I think that the risk of dying on any given day in a traffic accident is $p = 0.00001$, so most likely I would prefer any L_α lottery with $\alpha > 0.99999$ to L_1 as well. Let us now start decreasing α while posing the same question. Essentially, we are increasing the risk of the horrible outcome (death). I know that for very low values of α I will prefer to stay home. For instance, $\alpha = 0.90$ means that I am offered a choice between my best outcome ($500, life, package) with 90% and dying with 10% chance versus the outcome ($0, life, package). I know that I will stay home in this case, so this α is too low.

Obviously, somewhere between 0.99999 and 0.90, there exists a lottery that makes me precisely indifferent between the risky choice and staying home. (This is the assumption of *continuity*.) Suppose that in my case this turns out to be $\alpha = 0.9995$. That is, I am indifferent between staying home and getting my best outcome with probability 99.95% and dying with probability 0.05%. It should be quite clear at this point that your preferences might be very different here, and that they depend on your willingness to run risks. The upshot of this is that we can assign this special value of α to be the utility associated with the outcome o_1. In other words:

$$u(o_1) = 0.9995,$$

which, of course, also implies that $U(L_1) = 0.9995$. It is important to observe that the utility is thoroughly subjective and that it is not *derived* from any mathematical first principles. Instead, I was given a series of lotteries with varying values of α and at some point

I indicated which value made me indifferent between taking the outcome o_1 with certainty and a lottery that yields my best outcome with probability α and the worst outcome with complementary probability.

There is one remaining outcome, o_4, which is ($500, life, no package). Now, compared to o_1, this one leaves me without the package but it does give me $500. As I indicated before, the package is worth considerably less than $500 to me, which makes o_4 more attractive than o_1. Intuitively, then, the probability β at which I will be indifferent between L_4 and the compound lottery $\beta L_3 + (1-\beta)L_2$ should be *higher* than the probability α at which I am indifferent between L_1 and $\alpha L_3 + (1-\alpha)L_2$.

A bit of thought explains why: since the alternative L_4 is more attractive than L_1 (if $o_4 \succ o_1$, then it must be that $L_4 \succ L_1$), the compound lottery consisting of the best and worst outcomes has to be more attractive when I am indifferent between it and L_4 than when I am indifferent between it and L_1. Otherwise, my preferences would not be transitive. To see that, note that: $L_\beta \sim L_4 \succ L_1 \sim L_\alpha \Rightarrow L_\beta \succ L_\alpha$. If, instead, we suppose that $L_\alpha \succ L_\beta$, we would have:

$$
\begin{aligned}
L_\beta &\sim L_4 && \text{(by definition of } \beta\text{)} \\
L_\beta \sim L_4 \;\&\; L_4 \succ L_1 \Rightarrow L_\beta &\succ L_1 && \text{(by transitivity)} \\
L_\alpha &\succ L_\beta && \text{(by our assumption)} \\
L_\alpha \succ L_\beta \;\&\; L_\beta \succ L_1 \Rightarrow L_\alpha &\succ L_1 && \text{(by transitivity).}
\end{aligned}
$$

However, $L_\alpha \succ L_1$ contradicts $L_\alpha \sim L_1$ which is true by construction of L_α. Therefore, it must be the case that $L_\beta \succ L_\alpha$. Since in lotteries involving only the best and the worst outcomes, I always prefer those that yield the best outcome with higher probability, this implies that $\beta > \alpha$ as well. All of this confirms (mathematically) our intuition.

As before, I am offered a series of choices in order to determine β. As it so happens, $\beta = 0.9998$ makes me indifferent between L_4 and the lottery that gives me my best outcome with 99.98% and death with 0.02% chance. We can now assign o_4 this special value:

$$u(o_4) = 0.9998.$$

Again, note that these preferences are subjective and that the utilities we define this way depend on my risk attitude.

We now have the complete utility function:

$$
\begin{aligned}
u(o_1) &= 0.9995 & \qquad u(o_2) &= 0 \\
u(o_3) &= 1 & \qquad u(o_4) &= 0.9998.
\end{aligned}
$$

Let's see if it actually represents the preference ordering over the risky choice that I expressed. That is, is it true that $U(L_w) > U(L_h)$? Let us compute the expected utilities of the two lotteries:

$$
\begin{aligned}
U(L_w) &= pu(o_2) + (1-p)qu(o_3) + (1-p)(1-q)u(o_4) = 0.9998300016 \\
U(L_h) &= (1)u(o_1) = 0.9995,
\end{aligned}
$$

which clearly yields the result we expected. In other words, we have managed to assign utilities to the outcomes such that we can represent my preferences over risky choices with expected utilities: whenever I prefer a lottery L_a to lottery L_b, it will be the case that $U(L_a) > U(L_b)$. The converse is also true: if we compute the expected utilities of lotteries once we've assigned the utilities to certain outcomes, every time we find that $U(L_c) > U(L_d)$, it will also mean that $L_c \succ L_d$. In other words, the expected utility function *represents* my preferences.

This method of assigning utilities to represent preferences actually preserves the *intensity* of preferences in addition to risk attitudes. Intuitively, this is so because when I am asked whether I am indifferent between some certain outcome and the lottery between the best and the worst outcomes, I will take into account *how much* I care about the outcomes, not only whether I prefer one to the other. The value of β is not just greater than the value of α, it is greater by a *precise* amount.

In the example above, suppose I have an arrangement that allows me to stay home at a quarter my salary. This changes the "stay home" choice: I live, get \$125, and get my package. Let's denote this by o_1', or (\$125, life, package). Clearly this is more attractive to me than the original option: $o_1' \succ o_1$. What if I were offered half my salary when I stay home? Since this outcome, o_1'', gets me \$250, I would prefer it to both other options: $o_1'' \succ o_1' \succ o_1$. After some Q&A, I discover the probabilities that make me indifferent between the degenerate lotteries involving each of the new outcomes and the lottery involving the best and worst outcomes:

$$u(o_1') = 0.9997 \qquad \text{and} \qquad u(o_1'') = 0.9999.$$

Note in particular, that it matters how much I care about money in the sense that when I am not killed, I prefer \$500 without the package than \$125 with the package ($u(o_4) > u(o_1')$), but I prefer \$250 with the package to \$500 without the package ($u(o_1'') > u(o_4)$). Interestingly, it seems that I value the package somewhere between \$250 and \$375: that's because my preferences indicate that I would lose the package to secure \$375 and would lose \$250 to secure the package.

Observe now that since we also have $U(L_1') = 0.9997$ and $U(L_1'') = 0.9999$, and because the expected utility function represents my preferences, it follows that it has to be the case that I prefer the gamble of going to work to staying home at \$125 with the package, but that I also prefer staying at home with \$250 and the package to the gamble of going to work. This is hardly surprising: we just found out that the package is worth more than \$250 to me, so I would not prefer to go to work to get \$250 when it also means running a risk of dying and a risk of not getting the package even if I survive.

The precise amount of additional risk, however, does matter. Suppose the likelihood of UPS coming after work went up to, say, 75%. This now makes going to work a less risky choice because I am more likely to get the package compared to the original lottery. We now have:

$$U(L_w') = 0.9999400005,$$

which indicates that I would be willing to go back to work: $U(L_w') > U(L_1'')$.

If I were to express a preference for L_1'' over L_w', then my preferences would not be representable by an expected utility function. To see this suppose I expressed such a preference and that my preferences are representable by an expected utility function. Observe now that by construction my preference must be: $L_1'' \sim 0.9999L_3 + 0.0001L_2$, and $L_w' \sim 0.9999400005L_3 + 0.0000599995L_2$. But $L_1'' \succ L_w'$ then implies that:

$$\begin{aligned}
0.9999L_3 + 0.0001L_2 &\succ 0.9999400005L_3 + 0.0000599995L_2 \\
0.0001L_2 &\succ 0.0000400005L_3 + 0.0000599995L_2 \\
0.0000400005L_2 &\succ 0.0000400005L_3 \\
L_2 &\succ L_3.
\end{aligned}$$

This is clearly a contradiction because it means that the worst outcome is preferred to the best outcome.

3.5 Expected Utility as Useful Fiction

It is worth emphasizing that *utilities are fictitious numbers* and that under the assumptions of rationality, continuity, and independence, we can assign utilities to certain outcomes in a way that allows us to represent the preference ordering over risky gambles when we compute the expected utilities of lotteries. For any two lotteries, the expected utility of the preferred lottery will be higher. When we say than some choice yields the highest expected utility, we mean that this is the most preferred choice. It yields the highest expected utility because it is the most preferred choice. It is not most preferred because it yields the highest expected utility. Expected utilities represent preferences, but preferences are not determined by expected utility calculations. An individual who maximizes expected utility is one who chooses the action corresponding to the risky gamble he likes best. Provided that his preferences are representable with an expected utility function, we can then compute expected utilities from various actions to *infer* which gamble he likes best. If his preferences are not representable in this way, we cannot use expected utilities to make inferences.

We can thus dispense with the preferences over risky choices and instead use expected utility functions (provided we are willing to grant Axioms R, C, and I). But you absolutely must remember that when we are comparing the expected utilities of two risky choices, we are actually comparing the preferences over these choices. **It is *not* the case that the decision-maker prefers one risky choice over another because it yields the higher expected utility. Rather, that risky choice yields a higher expected utility *because it is preferred to the other*. Decision-makers do not have utilities, they have preferences. Utilities are only representations of these basic preferences. In essence, decision-makers behave *as if* they are maximizing expected utility when in fact they are choosing on the basis of their preferences.**

This is not to say that there are no valid and serious criticisms of expected utility theory (EUT). Invariably, they all come from experimental research that reveals that people do not, in fact, act like we would expect them to if their preferences were represented by a function with the expected utility form. As the Allais Paradox shows, one possibility is that people do not have an intuitive feel for probabilities so they do not recognize independence when they deal with compound lotteries. There are several alternatives to EUT, and most of these attempt to dispense with Axiom I.

Another problem is that the way people interpret information (and therefore how they act on it) seems to be affected by how this information is presented to them (framing). People seem to exhibit a stronger aversion to losses than interest in gains, so the may be more willing to run more risks to avoid some loss than to obtain a commensurate gain. You should read more about Kahneman-Tversky Prospect Theory (PT) and recent developments in it. PT seems to be supported by numerous studies across disciplines. The problem is that nobody has figured out yet how to bring framing, among other things, into our modeling world. It appears very subjective and has proven quite resistant to disciplined representation. As such, PT has had a limited impact on formal modeling despite its great promise.

Finally, people seem to exhibit an aversion to radical uncertainty, which is a type of uncertainty that is impossible to quantify. For instance, I might be uncertain whether riding the scooter to work would end in me arriving safely there. I could quantify this uncertainty using my knowledge of the chances of getting into an accident on the road I am using, the road conditions, and my riding ability. When I say I believe there is a 1/100th of a percent chance of an accident, I am expressing a subjective quantification that allows me to determine the risk associated with the choice of transportation. It might well be the case that I would ride the scooter except on rainy days because I believe I am not good enough on two wheels when the road is wet (not to mention other drivers).

There are, however, occasions when it is impossible to quantify uncertainty in any objective fashion. A simple illustration of that phenomenon is the so-called **Ellsberg Paradox**, which is usually taken as evidence that people exhibit aversion to the ambiguity that results from these unquantifiable uncertainties. The situation can be stated as follows. Suppose there is a box with 90 balls, 30 of which are red, and the rest either black or yellow. You do not know how many of these 60 balls are black and how many are yellow. You are given two sets of lottery pairs and asked to choose one lottery from each set. The monetary payoff depends on the color of a randomly drawn ball from the box, and the two sets are:

L_1	\$100 if Red, \$0 if Black or Yellow	L_1'	\$0 if Red or Yellow, \$100 if Black
L_2	\$100 if Red or Yellow, \$0 if Black	L_2'	\$0 if Red, \$100 if Black or Yellow

Table 1: The Ellsberg Paradox.

You prefer more money to less, and you are asked to express a preference between L_1 and L_1', and then between L_2 and L_2'. Many people report $L_1 \succ L_1'$ and $L_2' \succ L_2$. The logic seems to go as follows: L_1 gives me \$100 with $1/3$ probability, whereas in L_1' I may well end up with nothing since I have no idea how many black balls there. So I take L_1 over L_1'. In the second pair, L_2' guarantees me \$100 with probability $2/3$, whereas L_2 guarantees me that with only $1/3$ (it could be more but there's no way of knowing since I don't know the number of yellow balls). Therefore, I take L_2' over L_2.

So it sort of seems to make sense. Unfortunately, these preferences cannot be represented by expected utilities. To see this, observe that if you prefer $L_1 \succ L_1'$, it must be because subjectively you believe that there are fewer than 30 black balls, which implies you think there are more than 30 yellow ones. That is, you must believe that drawing a black ball is less likely than drawing a yellow ball. Analogously, if you prefer $L_2' \succ L_2$, it must be because you believe that there are fewer than 30 yellow balls, which implies you think there are more than 30 black ones. That is, you must believe that drawing a yellow ball is less likely than drawing a black ball. But you cannot believe that the number of black balls is simultaneously greater than and less than the number of yellow balls. The only consistent preferences are $L_1 \succ L'_1$ and $L_2 \succ L'_2$, or $L'_1 \succ L_1$ and $L'_2 \succ L_2$.

Not surprisingly, the preferences $L_1 \succ L'_1$ and $L'_2 \succ L_2$ cannot be represented with any expected utility function because they are not consistent in that way. To see that, let $r = \Pr(\text{Red})$ and $b = \Pr(\text{Black})$. Then, if the preferences are representable, it must be the case that $L_1 \succ L'_1$ if, and only if,

$$U(L_1) = ru(\$100) + (1-r)u(\$0) > U(L'_1) = (1-b)u(\$0) + bu(\$100)$$
$$r[u(\$100) - u(\$0)] + u(\$0) > b[u(\$100) - u(\$0)] + u(\$0)$$

and since we assumed you care about money, $u(\$100) > u(\$0)$, this reduces to:

$$r > b$$

Analogously, if preferences are representable, it must be the case that $L'_2 \succ L_2$ if, and only if,

$$U(L'_2) = ru(\$0) + (1-r)u(\$100) > U(L_2) = (1-b)u(\$100) + bu(\$0)$$
$$r[u(\$0) - u(\$100)] + u(\$100) > b[u(\$0) - u(\$100)] + u(\$100)$$
$$b[u(\$100) - u(\$0)] > r[u(\$100) - u(\$0)]$$
$$b > r$$

This establishes a contradiction, so these preferences cannot be represented with a function that has the expected utility form. Observe that the result holds no matter what utilities one attaches to monetary payoffs: we assumed you prefer more money to less, but the contradiction would still obtain if you preferred less money to more.

This is a situation in which the risk if actually not quantifiable: since we do not know how many black and yellow balls there are, the reasoning I outlined above that justifies the inconsistent preferences is a bit specious. The problem is that L_1 with its "certainty" of getting \$100 with probability $1/3$, appears much less vague than L'_1, where one simply has no idea what the odds are of getting \$100, only that they can be anywhere from 0 to $2/3$. In the second set, L'_2 with its "certainty" of getting \$100 with probability $2/3$, appears much less vague than L_2, where one can only say that the odds of getting \$100 are somewhere between $1/3$ and 1. As it happens, the Ellsberg Paradox has been shown to appear only when people have to express preference between some ambiguous choice and another, less vague, one. This is why it is usually taken as evidence of ambiguity aversion. Expected utility theory cannot represent preference over risky choices where the comparisons involve ambiguity aversion.

We have now seen instances of preferences that appear reasonable but that cannot be represented by functions with the expected utility form. It could be because they violate the independence axiom, or because they exhibit ambiguity aversion, or because they are sensitive to the framing of the choice, or it could be some other reason.[6]

[6]Recall that some lexicographic preferences cannot be represented with a utility function.

4 Risk Aversion

Now that we know that we can represent preferences with real numbers, what else can we say about them that might be helpful? In many settings, individuals seem to display aversion to risk. Let us formalize this notion in terms of lotteries.

Suppose $X \subset \mathbb{R}$. That is, the outcome is a continuous random variable on some subset of the real line. Although our results thus far derived the expected utility representation for the set of finite outcomes, they can be extended to the infinite domain at the expense of technical complications in the proofs (which is why we did not do it). So, suppose outcome is a continuous random variable x. You could think of x in terms of money. Although not necessary, it will be helpful to understand the results. Let $u(\cdot)$ be a utility function defined on sure amounts of money. This is different from the vNM utility function $U(\cdot)$ which is defined on lotteries. We shall call $u(\cdot)$ the *Bernoulli utility function*.

For any lottery p, let p_E represent the expected value of p, or:[7]

$$p_E = \sum_X p(x)x.$$

That is, the expected value of a lottery is the outcome (real number) multiplied by the probability that it occurs assigned to it by the lottery, summed over all possible outcomes. The definition in the continuous case is analogous. Note that this does not use $u(x)$ but the value of the outcome x, which is a real number. To see the difference, note that:

$$U(p) = \sum_X p(x)u(x).$$

That is, while u is defined on outcomes, U is defined on lotteries over these outcomes. Note that p_E, like $U(p)$, is a real number.

For any outcome x, let the lottery $\delta(x)$ be the simple lottery that yields x with certainty. For each lottery p then, $\delta(p_E)$ then is the degenerate lottery that yields the expected value of the lottery p for sure. The attitude toward risk relates the utility of getting a sure-thing outcome (i.e. an outcome from such a degenerate lottery) to taking a risky gamble of the original lottery.

The expected value of a lottery p, p_E, is a real number, and $u(p_E)$ is the utility associated with this number. To see this, note that $U(\delta(p_E)) = (1)u(p_E) = u(p_E)$. We shall compare this utility with the expected utility of the lottery: $U(p)$. Here is the main result which we shall not prove.

PROPOSITION 5. *For all lotteries $p \in \mathcal{P}$, it is the case that $\delta(p_E) \succeq p$ if, and only if, $u(\cdot)$ is concave; it is the case that $p \succeq \delta(p_E)$ if, and only if, u is convex; and it is the case that $\delta(p_E) \sim p$ if, and only if, u is linear.* □

A player who prefers to get the expected value of a lottery for sure instead of taking a risky gamble is said to be *risk averse*. The lack of willingness to take a risky gamble implies a concave utility function $u(\cdot)$ because it requires that $u(p_E) > U(p)$. An example of a concave utility function is $u(x) = \ln(x)$.

[7] This is the discrete case, but the definition of the continuous case is analogous.

If a player prefers to take the risky gamble instead of the expected value of the lottery for sure, then he is *risk acceptant*. The willingness to take a risky gamble implies a convex utility function $u(\cdot)$ because it requires that $u(p_E) < U(p)$. An example of a convex utility function is $u(x) = x^2$.

Finally, a player is *risk neutral* if he is indifferent between the sure thing and the risky gamble. This implies a linear utility function $u(\cdot)$ because it requires that $u(p_E) = U(p)$. An example of a linear utility function is $u(x) = x$.

The three risk attitudes are shown in Figure 4.[8] Consider the concave utility function *(Risk Averse)* and the two outcomes x and y. Take some $\alpha \in (0, 1)$ and consider the lottery $p = \alpha\delta(x) + (1-\alpha)\delta(y)$. That is, an α chance of x and an $1-\alpha$ chance of y. The expected value of this lottery is then $p_E = \alpha x + (1-\alpha)y$.

What does the player prefer: the sure thing $\delta(p_E)$ or the lottery p? To decide, compare the expected utilities of the two. For $\delta(p_E)$, the utility is

$$U(\delta(p_E)) = u(p_E) = u(\alpha x + (1-\alpha)y),$$

while the expected utility of p is

$$U(p) = \alpha u(x) + (1-\alpha)u(y).$$

Because the function $u(\cdot)$ is concave, $u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y)$. Thus, the player must be risk averse because the expected utility of the sure thing exceeds the expected utility of the risky gamble. That is, this player must prefer to take the expected value of the lottery than play the lottery.

Conversely, a risk acceptant player prefers to gamble than get the expected outcome of the gamble. Correspondingly, the expected utility of taking the expected value of the lottery with certainty must be less than the expected utility of playing the lottery.

A risk neutral player is indifferent between the two.

To illustrate this with an example, consider the interval $X = [-1, 1]$, and a player with the linear utility function $u(x) = ax + b$, with $a > 0$ (so we get the strictly increasing property discussed above). Does this player prefer getting .5 for sure or does he prefer to gamble on the lottery p, in which $p(-1) = .25$ and $p(1) = .75$?

$$\begin{aligned} u(.5) &= .5a + b \\ U(p) &= (.25)[(-1)a + b] + (.75)[(1)a + b] = .5a + b \end{aligned}$$

So this utility function represents a player that is indifferent between the sure thing and the gamble on p. As we noticed above, players whose preferences are represented by linear utility functions are risk neutral.

Consider now the utility function $u(x) = (ax + b)^r$ and two players, player A, with $r = 1/2$ and player B, with $r = 2$. Let's examine the same gamble on p as before and let, for simplicity, $a = b = 2$.

$$\begin{aligned} u_A(.5) &= \sqrt{.5a + b} = \sqrt{3} \\ U_A(p) &= (.25)\sqrt{-a + b} + (.75)\sqrt{a + b} = 1.5 \end{aligned}$$

[8] The figure may be a little misleading because it gives the impression that x and y must be such that $u(x)$ and $u(y)$ lie exactly on the 45 degree line. This is not the case: pick any x and y and connect them with a straight line. By concativity, the graph of the risk averse utility function will lie above this segment.

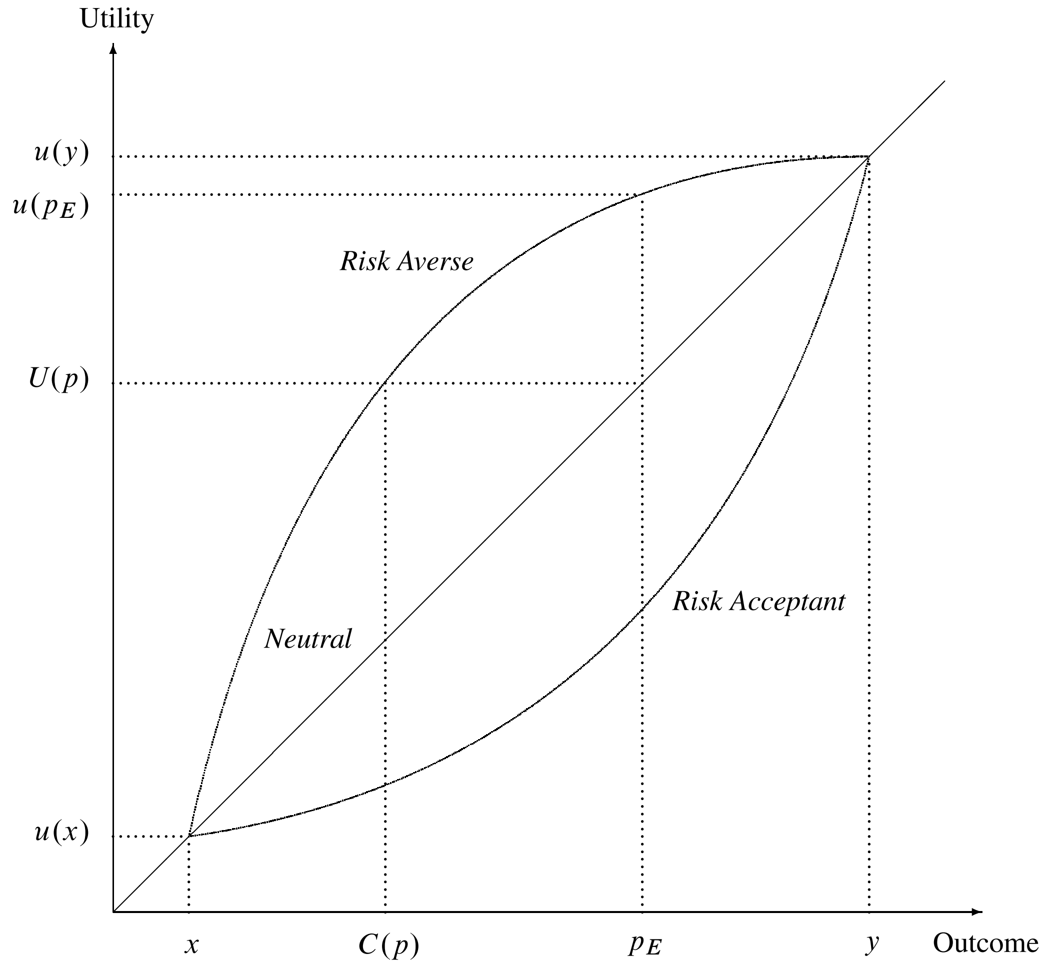


Figure 4: Concativity, Risk Aversion, and Certainty Equivalents. Denote the expected utility of p by $U(p) = \alpha u(x) + (1-\alpha)u(y)$; the expected value of p by $p_E = \alpha x + (1-\alpha)y$; the certainty equivalent of p by $C(p)$; and the utility of the expected value of p by $u(p_E)$.

Since $\sqrt{3} > 1.5$, player A must be preferring the sure thing to the gamble. That is, this player is risk averse. It is not difficult to show that u_A is concave.

$$u_B(.5) = (.5a + b)^2 = 9$$
$$U_B(p) = (.25)(-a + b)^2 + (.75)(a + b)^2 = 12$$

In this case, player B must be preferring the gamble to the sure thing (since $12 > 9$). That is, this player is risk acceptant. Again, it is not hard to show that u_B is convex.

More generally, for any $r < 1$, the utility function $u(x) = (ax + b)^r$ will be concave and thus useful to represent a risk averse player. For any $r > 1$, the utility function will be convex and thus useful to represent a risk acceptant player. For $r = 1$, the utility function is linear and can be used to represent a risk neutral player.

Sometimes it will be useful to find an outcome, whose utility is the same as the expected utility of some lottery. In other words, we go backwards: we start with some utility and we wish to find the outcome that produces it. Consider two outcomes $x, y \in X$ and the utility function $u : X \to \mathbb{R}$. Let p be some lottery such that x occurs with probability α and y occurs with probability $1 - \alpha$. From the intermediate value theorem of calculus, we know that for every $\alpha \in [0, 1]$, there is some value z with $u(z) = \alpha u(x) + (1 - \alpha)u(y) = U(p)$. From the expected utility representation, we know that for any such z, it is the case that $\delta(z) \sim p$. To see this, note that $U(\delta(z)) = u(z) = U(p)$. This z is called the *certainty equivalent* of p.

Almost all utility functions that we shall deal with in this course will be continuous and strictly increasing. The following proposition then gives us a useful result that states that this certainty equivalent always exists and is unique.

PROPOSITION 6. *If $u(\cdot)$ is continuous and strictly increasing, and if X is an interval of $\mathbb{R}$, then every $p \in \mathcal{P}$ has a unique certainty equivalent.* □

Referring back to Figure 4, the certainty equivalent of p, denoted by $C(p)$, yields the same utility as the expected utility of p. Since the player is risk averse, $C(p) \leq p_E$.